

Principal actions of stacky Lie groupoids

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Abstract

Stacky Lie groupoids are generalizations of Lie groupoids in which the “space of arrows” of the groupoid is a differentiable stack. In this paper, we consider actions of stacky Lie groupoids on differentiable stacks and their associated quotients. We provide a characterization of *principal actions* of stacky Lie groupoids, i.e., actions whose quotients are again differentiable stacks in such a way that the projection onto the quotient is a principal bundle. As an application, we extend the notion of Morita equivalence of Lie groupoids to the realm of stacky Lie groupoids, providing examples that naturally arise from non-integrable Lie algebroids.

Contents

1	Introduction	2
2	Preliminaries	4
2.1	Categories fibred in groupoids	5
2.2	Differentiable stacks	7
2.3	Morphisms of differentiable stacks	9
2.4	Lie groupoids and Hilsum-Skandalis maps	11

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3	Stacky Lie groupoids and actions	15
3.1	Stacky Lie groupoids	15
3.2	Actions	21
3.3	Equivariant morphisms	23
3.4	Stacky principal bundles	24
4	Quotients	29
4.1	The (pre)quotient of an action	29
4.2	Conditions for the prequotient to be a prestack	31
4.3	The universal property of (pre)quotients	34
4.4	Equivariant maps and (pre)quotients	37
4.5	Prequotients and the action-projection map	39
5	Principal actions of stacky Lie groupoids	42
5.1	The main theorem: characterization of principal actions	42
5.2	Proofs of lemmas	44
5.3	Proof of Proposition 5.5	47
5.4	Application: tensor product of stacky bundles	52
6	Morita equivalence of stacky Lie groupoids	53
6.1	Stacky bibundles and Morita equivalence	54
6.2	Example from non-integrable Lie algebroids	55
6.3	Properties of Morita equivalence	58
A	Appendices	66
A.1	Proofs of properties of submersions	66
A.2	Inversion and actions	68
A.3	The prequotient construction	71
A.4	Quotient in stages	73

1 Introduction

Lie groupoids have widespread use in several areas of mathematics, and in recent years some of their “higher” versions have drawn much attention, particularly in the study of higher categorical structures and higher gauge theory, see e.g. [2, 3, 4, 8, 10, 18, 33, 58]. This paper concerns *stacky Lie groupoids*, which are generalizations of Lie groupoids $\mathcal{G} \rightrightarrows M$ where \mathcal{G} is allowed to be a differentiable stack, while M is a smooth manifold. These objects were introduced in [49] in connection with the problem of integrating Lie algebroids.

Recall that, while any finite-dimensional Lie algebra can be integrated to a Lie group, not every Lie algebroid integrates to a Lie groupoid, see [15] and references therein. Whenever a Lie algebroid is integrable, the so-called “path-space construction” [13, 15, 45] provides a concrete way to obtain an integrating Lie groupoid. However, for a non-integrable Lie algebroid this construction only leads to a *topological* groupoid. The starting point in [49] is the observation (see also [53]) that the topological groupoids arising in this way naturally carry the structure of *differentiable stacks*, and that this additional information allows one to recover the underlying Lie-algebroid data; as a consequence, [49] establishes a correspondence between (arbitrary)

Lie algebroids and étale stacky Lie groupoids, which places the usual correspondence between integrable Lie algebroids and Lie groupoids in a broader framework.

The main purpose of this paper is to study actions of stacky Lie groupoids on differentiable stacks, with focus on the notion of *principality*, inspired by the usual study of actions of Lie groupoids on manifolds. To put our results in context, recall that when a Lie groupoid G acts on a manifold X , the orbit space X/G generally fails to be a smooth manifold. An action is called *principal* if X/G is a manifold and, additionally, the quotient map $X \rightarrow X/G$ makes X into a principal G -bundle with base X/G . A classical result in geometry asserts that an action is principal if and only if it is free and proper, and this completely describes G -actions that correspond to principal bundles in the smooth category¹. In this paper, we are concerned with the analogous issue in the realm of differentiable stacks; more precisely, we answer the following question: in the context of stacky Lie groupoids acting on differentiable stacks, which conditions characterize actions that give rise to (stacky) principal bundles?

More specifically, given an action of a stacky Lie groupoid \mathcal{G} on a differentiable stack \mathcal{X} , we first address the construction of the “orbit space” \mathcal{X}/\mathcal{G} , building on [8]. Our main result (Theorem 5.2) then provides a complete characterization of actions for which this quotient inherits the structure of a differentiable stack in such a way that the natural projection $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}$ defines a principal \mathcal{G} -bundle. This characterization is given in terms of a simple “weak representability” condition (see Definition 2.6) for the \mathcal{G} -action. This condition turns out to be automatically verified if \mathcal{G} is an ordinary Lie groupoid (see Corollary 5.3), which implies the well-known fact that, in the broader context of differentiable stacks, *any* smooth action of a Lie groupoid G on a manifold X is principal; indeed, the quotient stack $[X/G]$ is always differentiable, and X is a principal G -bundle over $[X/G]$. Just as in the smooth setting, our characterization of principal actions of stacky Lie groupoids provides a complementary approach to other existing viewpoints to higher principal bundles, as found e.g. in [3, 4, 54].

In the classical theory of Lie groupoids and its various applications, a central role is played by the notion of *Morita equivalence* (see e.g. [6, 34, 35]). One of the applications of our results is an extension of this notion to the realm of stacky Lie groupoids. Indeed, a common approach to Morita equivalence of Lie groupoids uses principal bundles to define “generalized morphisms”, in such a way that Morita equivalence is expressed by *biprincipal bibundles* (see e.g. [35, Sec. 2]). Using (bi)principal (bi)bundles of stacky Lie groupoids, one may define Morita equivalence analogously, and our characterization of principal actions is key in showing that stacky principal bibundles can be “composed”, which ensures that Morita equivalence is an equivalence relation amongst stacky Lie groupoids. We remark that another approach to categorified bibundles and Morita equivalence of 2-groupoids through simplicial methods is developed in [28, Sec. 6]; for strict 2-groupoids, another viewpoint to Morita equivalence can be found in [21].

Much of our motivation for this work comes from Poisson geometry, where the existing notion of Morita equivalence [56, 57] only applies to *integrable* Poisson manifolds, i.e., Poisson manifolds whose underlying Lie algebroids are integrable. A possible approach to Morita equivalence of non-integrable Poisson manifolds via stacks was suggested in [11, Section 9.3] (see also [12]), and this paper may be regarded as the first foundational step in this direction. Indeed, in light of the correspondence between Lie algebroids and stacky Lie groupoids in [49], one may expect to have a description of Morita equivalence of stacky Lie groupoids only in terms of Lie-algebroid data; this could then be specialized to the case of Poisson manifolds (see [50]). We plan to address these issues in subsequent work.

¹The characterization of principal actions of groupoids is more subtle if one considers infinite-dimensional manifolds, see [32] for a thorough discussion.

For the reader’s convenience, we outline the structure and content of the paper:

- Section 2 recalls the main definitions concerning differentiable stacks and Lie groupoids and collects some technical results used in the sequel. An important concept introduced in this section is that of *weak representability* of a morphism of differentiable stacks (Def. 2.6), which plays a key role in the study of principal actions.
- Section 3 recalls the notion of stacky Lie groupoid and introduces actions of stacky Lie groupoids on (differentiable) stacks, pointing out their key features. The notion of *principal bundle* for stacky Lie groupoids is also discussed in this section (Def. 3.24), along with some of its basic properties and examples.
- In Section 4, we define the “quotient space” associated with an action of a stacky Lie groupoid $\mathcal{G} \rightrightarrows M$ on a differentiable stack \mathcal{X} . This quotient is initially defined as a category fibred in groupoids, referred to as the “prequotient” (see Prop. 4.1); its stackification, denoted by \mathcal{X}/\mathcal{G} , is our object of interest. The main properties of (pre)quotients are presented in this section.
- Section 5 contains the main result of the paper: Theorem 5.2, which provides a characterization of principal actions of stacky Lie groupoids, i.e., it gives a necessary and sufficient condition ensuring that an action of a stacky Lie groupoid \mathcal{G} on a differentiable stack \mathcal{X} gives rise to a principal bundle $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}$. As an application, we show that the usual “composition” (or “tensor product”) of principal bundles of Lie groupoids naturally extends to stacky Lie groupoids (Prop. 5.13).
- In Section 6, following the usual theory of Lie groupoids, we consider biprincipal bibundles, i.e., differentiable stacks carrying commuting principal actions of two stacky Lie groupoids, one on the right and the other on the left. These are the central objects for the definition of *Morita equivalence* of stacky Lie groupoids. We present a concrete example of Morita equivalence arising from a non-integrable transitive Lie algebroid that generalizes the usual Atiyah algebroid associated with a principal S^1 -bundle. We verify two key properties of our extended notion of Morita equivalence: that it is an equivalence relation (Thm. 6.10), and that it recovers the classical one when restricted to Lie groupoids (Prop. 6.20).

The appendices, organized in four sections, collect some technical material, including proofs of auxiliary results needed throughout the paper.

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2 Preliminaries

In this section, we collect basic facts about stacks used in the sequel of the paper. Stacks have been extensively studied in algebraic geometry, see e.g. [1, 17, 26, 51, 52]; more recently, there

has been an increasing interest in stacks in the categories of topological spaces and smooth manifolds, see e.g. [6, 31, 40]. This paper focuses on stacks in the differentiable category, in the spirit of [6, 31], where details and proofs omitted here can be found; see also Remark 2.5.

Before moving on, we set up some notation. Given a category \mathcal{X} and an object x of \mathcal{X} , we will use either the notation $x \in \text{Obj}(\mathcal{X})$, or simply $x \in \mathcal{X}$. We denote the set of morphisms from x to y by $\text{Hom}_{\mathcal{X}}(x, y)$. For 2-categories we will often write compositions of 2-morphisms as follows: *horizontal compositions* are denoted by ‘ \circ ’ or juxtaposition (the same notation will be used for 1-morphisms), and *vertical compositions* by ‘ $*$ ’. For example, if A, B, C are objects in a 2-category, and a, b, c, d are 1-morphisms, and α and β are 2-morphisms, we write $b \circ a = ba$, $\beta \circ \alpha = \beta\alpha$ and $\beta * \alpha$ for the compositions depicted in the diagrams below:

$$\begin{array}{ccc} \begin{array}{ccc} A & \begin{array}{c} \xrightarrow{a} \\ \downarrow \alpha \\ \xrightarrow{c} \end{array} & B \\ & \downarrow \beta & \\ C & \begin{array}{c} \xrightarrow{b} \\ \downarrow \beta \\ \xrightarrow{d} \end{array} & \end{array} & \mapsto & \begin{array}{ccc} A & \begin{array}{c} \xrightarrow{ba} \\ \downarrow \beta\alpha \\ \xrightarrow{dc} \end{array} & C \end{array} \\ \\ \begin{array}{ccc} A & \begin{array}{c} \xrightarrow{a} \\ \downarrow \alpha \\ \xrightarrow{b} \\ \downarrow \beta \\ \xrightarrow{c} \end{array} & B \end{array} & \mapsto & \begin{array}{ccc} A & \begin{array}{c} \xrightarrow{a} \\ \downarrow \beta * \alpha \\ \xrightarrow{c} \end{array} & B \end{array} \end{array}$$

For identity morphisms and 2-morphisms we will use the notation ‘id’ as follows:

$$A \xrightarrow{\text{id}_A} A \qquad \begin{array}{ccc} A & \begin{array}{c} \xrightarrow{a} \\ \downarrow \text{id}_a \\ \xrightarrow{a} \end{array} & B \end{array}$$

A square

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ c \downarrow & & \downarrow b \\ C & \xrightarrow{d} & D \end{array}$$

is called *2-commutative* if there is a given 2-isomorphism $\alpha : dc \rightarrow ba$, in which case we say that the square is 2-commutative *with respect to* α .

2.1 Categories fibred in groupoids

Let \mathcal{C} be the category of smooth manifolds². We endow \mathcal{C} with the Grothendieck topology given by open covers. We recall the definition of the (strict) 2-category of categories fibred in groupoids over \mathcal{C} , denoted by $\text{CFG}_{\mathcal{C}}$. A **category fibred in groupoids** over \mathcal{C} , i.e., an object in $\text{CFG}_{\mathcal{C}}$, is a pair (\mathcal{X}, π) , where \mathcal{X} is a category and $\pi : \mathcal{X} \rightarrow \mathcal{C}$ is a functor, satisfying the following conditions:

- (i) Any diagram

$$\begin{array}{ccc} & & y \\ & & \downarrow \\ U & \xrightarrow{f} & V \end{array}$$

²Manifolds are not necessarily assumed to be Hausdorff, as this property fails in many natural examples of spaces of arrows of Lie groupoids.

can be completed to a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{a} & y \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & V \end{array}$$

where $f : U \rightarrow V$ is a morphism in \mathcal{C} , $a : x \rightarrow y$ is a morphism in \mathcal{X} , the vertical lines mean that $\pi(x) = U$ and $\pi(y) = V$, and the commutativity means that $\pi(a) = f$.

- (ii) Any morphism $a : x \rightarrow y$ in \mathcal{X} is cartesian, i.e., for any commutative diagram of solid arrows as below,

$$\begin{array}{ccccc} & & b & & \\ & \searrow & \curvearrowright & \nearrow & \\ z & \xrightarrow{c} & x & \xrightarrow{a} & y \\ \downarrow & & \downarrow & & \downarrow \\ W & \xrightarrow{g} & U & \xrightarrow{f} & V \end{array}$$

(i.e., $\pi(a) = f$ and $\pi(b) = fg$), there exists a unique c that makes the diagram commute (i.e., $ac = b$ and $\pi(c) = g$).

If there is no risk of confusion, we simplify notation and denote a category fibred in groupoids (\mathcal{X}, π) simply by \mathcal{X} . We may also use the notation $\pi_{\mathcal{X}}$ for π if \mathcal{X} is not clear from the context.

If \mathcal{X} is a category fibred in groupoids and U is a manifold, we define the **fiber** of \mathcal{X} over U , denoted by either $\mathcal{X}(U)$ or \mathcal{X}_U , as the category whose objects are the objects x of \mathcal{X} that lie over U (i.e., $\pi(x) = U$), and whose morphisms $a : x \rightarrow y$ in \mathcal{X}_U are those in \mathcal{X} that lie over the identity of U (i.e., $\pi(a) = \text{id}_U$). Conditions (i) and (ii) above imply that the fibers of π over any object of \mathcal{C} are groupoids (i.e., categories in which all the morphisms are invertible). When $a : x \rightarrow y$ is a cartesian arrow, with $f = \pi(a) : U \rightarrow V$, we refer to x as the **pullback** of y by f . (Note that x is uniquely defined, up to canonical isomorphism, by y and f .) We use the notation $y|_U$ or f^*y for x . Given a morphism $b : y' \rightarrow y$ over id_V , there is an induced morphism $f^*b = b|_U : f^*y' \rightarrow f^*y$.

A **morphism** between categories fibred in groupoids (\mathcal{X}_1, π_1) and (\mathcal{X}_2, π_2) is a functor

$$F : \mathcal{X}_1 \rightarrow \mathcal{X}_2$$

such that $\pi_2 F = \pi_1$. The **2-morphisms** between $F, F' : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ are the natural transformations $\eta : F \rightarrow F'$ such that $\pi_2(\eta(x) : F(x) \rightarrow F'(x)) = \text{id}_{\pi_1(x)}$ for any object $x \in \mathcal{X}_1$. We recall that any 2-morphism in $\text{CFG}_{\mathcal{C}}$ is an isomorphism with respect to vertical composition.

Two categories fibred in groupoids \mathcal{X} and \mathcal{Y} are **isomorphic** if there are morphisms $F : \mathcal{X} \rightarrow \mathcal{Y}$ and $F' : \mathcal{Y} \rightarrow \mathcal{X}$ such that the compositions FF' and $F'F$ are isomorphic to the corresponding identities. We recall that F is an isomorphism in this sense if and only if for any manifold U the restriction $F_U : \mathcal{X}_U \rightarrow \mathcal{Y}_U$ is an equivalence of categories.

Any manifold X naturally gives rise to a category fibred in groupoids, still denoted by X , whose fiber over a manifold U is given by $\text{Hom}(U, X)$. A category fibred in groupoids \mathcal{X} is **representable** if there is a manifold X (whose associated category fibred in groupoids is) isomorphic to it.

We also recall the **fibred product** of morphisms of categories fibred in groupoids. Let

$$F_i : \mathcal{X}_i \rightarrow \mathcal{Y}$$

be morphisms of categories fibred in groupoids, for $i = 1, 2$. The objects of the fibred product $\mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2$ (we will also use the notation $\mathcal{X}_1 \times_{F_1, \mathcal{Y}, F_2} \mathcal{X}_2$) are triples (x_1, a, x_2) with $x_i \in \mathcal{X}_i$ and $a : F_1(x_1) \rightarrow F_2(x_2)$, where x_1, x_2 are assumed to lie over the same manifold U , and a lies over id_U (hence it is an isomorphism). A morphism $(b_1, b_2) : (x_1, a, x_2) \rightarrow (x'_1, a', x'_2)$ is given by a pair of morphisms $b_i : x_i \rightarrow x'_i$ in \mathcal{X}_i , for $i = 1, 2$, such that the diagram

$$\begin{array}{ccc} F_1(x_1) & \xrightarrow{a} & F_2(x_2) \\ F_1(b_1) \downarrow & & \downarrow F_2(b_2) \\ F_1(x'_1) & \xrightarrow{a'} & F_2(x'_2) \end{array}$$

commutes.

We recall that a diagram (in $\text{CFG}_{\mathcal{C}}$)

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{X}_1 \\ F'_1 \downarrow & & \downarrow F_1 \\ \mathcal{X}_2 & \xrightarrow{F_2} & \mathcal{Y} \end{array} \quad (2.1)$$

is called **2-cartesian** if it is 2-commutative and the induced map from \mathcal{W} to $\mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2$ is an isomorphism. In this case, we refer to this square as a **pullback** diagram, and we say that F'_1 is the **base change** of F_1 by F_2 . (Occasionally, we may refer to 2-cartesian diagrams just as cartesian.)

A morphism $F : \mathcal{X} \rightarrow \mathcal{Y}$ of categories fibred in groupoids is said to be a **monomorphism** if, for any manifold U , the restriction $F_U : \mathcal{X}_U \rightarrow \mathcal{Y}_U$ of F over U is fully faithful. The morphism F is said to be an **epimorphism** if, for any $U \in \mathcal{C}$ and any $y \in \mathcal{Y}_U$, there exists a cover $(U_\alpha \rightarrow U)_\alpha$ and, for any α , there exists $x_\alpha \in \mathcal{X}_{U_\alpha}$ such that $F(x_\alpha) \simeq y|_{U_\alpha}$ in \mathcal{Y}_{U_α} . For a morphism $F : X \rightarrow Y$ between manifolds, the condition that F is an epimorphism is equivalent to the existence of local sections around any point of Y . We also recall that being an epimorphism (resp. monomorphism) is stable under composition and base change (i.e., in a 2-cartesian square (2.1), if F_1 has this property, then so does F'_1).

A morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of categories fibred in groupoids is called **representable** if, for any manifold Y and morphism $Y \rightarrow \mathcal{Y}$, the fibred product $Y \times_{\mathcal{Y}} \mathcal{X}$ is representable; this property is preserved under composition and base change. Given a morphism $X \rightarrow Y$ between manifolds, it is representable if and only if it is a submersion (see e.g. [31, Lem. 71]). As a consequence, for a representable morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of categories fibred in groupoids and a morphism $Y \rightarrow \mathcal{Y}$ from a manifold Y , the induced map $Y \times_{\mathcal{Y}} \mathcal{X} \rightarrow Y$ is automatically a submersion (so representable morphisms are also referred to as **representable submersions**). It also follows that $\mathcal{X} \rightarrow \mathcal{Y}$ is a representable epimorphism if and only if $Y \times_{\mathcal{Y}} \mathcal{X} \rightarrow Y$ is a surjective submersion of manifolds for all $Y \rightarrow \mathcal{Y}$, where Y is a manifold.

2.2 Differentiable stacks

Definition 2.1 *A category fibred in groupoids \mathcal{X} is called a **stack** if the following two conditions are satisfied:*

- (A1) *Given a manifold U and two objects x, y in \mathcal{X}_U , for every open cover $(U_\alpha \rightarrow U)_\alpha$, and for every collection of isomorphisms $\phi_\alpha : x|_{U_\alpha} \rightarrow y|_{U_\alpha}$ over U_α such that $\phi_\alpha|_{U_{\alpha\beta}} = \phi_\beta|_{U_{\alpha\beta}}$, there is a unique isomorphism $\phi : x \rightarrow y$ such that $\phi|_{U_\alpha} = \phi_\alpha$. (Here $U_{\alpha\beta} = U_\alpha \times_U U_\beta$.)*

(A2) Let U be a manifold and $(U_\alpha \rightarrow U)_\alpha$ be an open cover. Let x_α be an object in \mathcal{X}_{U_α} , and let $\phi_{\beta\alpha} : x_\alpha|_{U_{\alpha\beta}} \rightarrow x_\beta|_{U_{\alpha\beta}}$ be morphisms over $U_{\alpha\beta}$ satisfying $\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}$ (over $U_{\alpha\beta\gamma} = U_\alpha \times_U U_\beta \times_U U_\gamma$). Then there exist an object x over U , and isomorphisms $\phi_\alpha : x|_{U_\alpha} \rightarrow x_\alpha$ over U_α such that $\phi_\beta = \phi_{\beta\alpha} \circ \phi_\alpha$ (over $U_{\alpha\beta}$).

A category fibred in groupoids is a **prestack** if it satisfies (A1). Note that (A1) implies that x in (A2) is unique up to canonical isomorphism.

A morphism between stacks is just a morphism of the underlying categories fibred in groupoids. If $F : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of stacks, then the definition of epimorphism given in Section 2.1 is equivalent to the one given in [6, Def. 2.3].

We will need the notion of stackification of a category fibred in groupoids. We recall its main, well-known properties (see e.g. [31, Prop. 52, Lem. 53]):

Proposition 2.2 *Let \mathcal{X} be a category fibred in groupoids. Then there is a stack \mathcal{X}^\sharp , called the **stackification** of \mathcal{X} , and a morphism $\mathbf{b} : \mathcal{X} \rightarrow \mathcal{X}^\sharp$ such that that the following properties hold:*

- (i) *For any stack \mathcal{S} and morphism $F : \mathcal{X} \rightarrow \mathcal{S}$ there is a pair (F^\sharp, ζ) , where $F^\sharp : \mathcal{X}^\sharp \rightarrow \mathcal{S}$ and $\zeta : F^\sharp \mathbf{b} \xrightarrow{\sim} F$.*
- (ii) *Let \mathcal{S} be a stack, $F_i : \mathcal{X} \rightarrow \mathcal{S}$, $i = 1, 2$, be morphisms and $\eta : F_1 \xrightarrow{\sim} F_2$. Let $F_i^\sharp : \mathcal{X}^\sharp \rightarrow \mathcal{S}$ and $\zeta_i : F_i^\sharp \mathbf{b} \xrightarrow{\sim} F_i$, $i = 1, 2$, be as in (i). Then there exists a unique $\eta^\sharp : F_1^\sharp \xrightarrow{\sim} F_2^\sharp$ such that $\eta * \zeta_1 = \zeta_2 * (\eta^\sharp \circ \text{id}_{\mathbf{b}})$.*

Moreover,

- (iii) *If \mathcal{X} is a prestack then the stackification map $\mathbf{b} : \mathcal{X} \rightarrow \mathcal{X}^\sharp$ is a monomorphism and an epimorphism.*
- (iv) *Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of prestacks. Then, the stackified map $F^\sharp : \mathcal{X}^\sharp \rightarrow \mathcal{Y}^\sharp$ is an isomorphism if and only if F is a monomorphism and an epimorphism.*

Definition 2.3 *A stack \mathcal{X} is called **differentiable** if there exists a representable epimorphism $X \rightarrow \mathcal{X}$ from a manifold X . We call such a morphism an **atlas** of \mathcal{X} .*

We mention some standard but important examples.

Example 2.4

- (a) The category fibred in groupoids associated with a manifold X is a differentiable stack. The manifold X itself is an atlas.
- (b) If a Lie group G acts on a manifold X , then there is an associated *quotient stack* $[X/G]$, which is a differentiable stack for which X can be taken as an atlas. In particular, when X is a point, the quotient stack is called the **classifying space** of the Lie group G , and it is denoted by BG .
- (c) The definition of quotient stack can be generalized to the setting of Lie groupoids acting on manifolds. We will provide more details and references in Section 2.4.

Remark 2.5 We point out that there is a broader viewpoint to stacks through the theory of *higher stacks* developed in [29, 41, 46]. This theory unifies all levels of n -stacks, where n is a non-negative integer, or ∞ . In this hierarchy, a 0-stack is a sheaf. Recall that a presheaf in a category \mathcal{C} is a contravariant functor from \mathcal{C} to the category of sets. If the category is endowed with a Grothendieck topology, then one may define the so-called local isomorphisms in the category of presheaves on \mathcal{C} . Localizing with respect to local isomorphisms gives us the category of sheaves on \mathcal{C} . This way of defining sheaves, going back to [1], is totally internal; one arrives at the concept of sheaves without explicitly defining what they are.

For higher stacks, one passes from a category to a simplicial enriched category and from sets to simplicial sets. In this framework, one may consider geometric stacks [47], which extend what we call differentiable stacks in this paper; roughly, these are the higher stacks presentable by higher groupoids (see also [39]). Although the theory is mostly driven by algebraic geometric applications, the same ideas carry over to differential geometry, see e.g. [37] (and the theory in fact simplifies in the context of manifolds). From this perspective, a stack is the “stackification” (analogous to the localization in the case of sheaves) of a (higher) functor from the category of manifolds to the bicategory of groupoids. Such a functor may be explicitly expressed as a category fibred in groupoids, which is the viewpoint taken in this paper.

2.3 Morphisms of differentiable stacks

In the sequel, the following weak form of representability will be central:

Definition 2.6 *A morphism of differentiable stacks $F : \mathcal{X} \rightarrow \mathcal{Y}$ is called **weakly representable** if there exists an atlas $Y \rightarrow \mathcal{Y}$ such that the fibred product $\mathcal{X} \times_{\mathcal{Y}} Y$ is representable.*

Proposition 2.7 *A morphism $F : \mathcal{X} \rightarrow \mathcal{Y}$ is weakly representable if and only if for all representable morphisms $U \rightarrow \mathcal{Y}$, where U is a manifold, the fibred product $\mathcal{X} \times_{\mathcal{Y}} U$ is representable.*

PROOF: The ‘if’ part is clear since \mathcal{Y} is assumed to have an atlas. For the converse, take an atlas $Y \rightarrow \mathcal{Y}$ such that $\mathcal{X} \times_{\mathcal{Y}} Y$ is representable, and a representable map $U \rightarrow \mathcal{Y}$ from a manifold U . We have to show that $\mathcal{X} \times_{\mathcal{Y}} U$ is representable. Taking the fibred product $P = U \times_{\mathcal{Y}} Y$, the projection $P \rightarrow U$ is a surjective submersion; moreover, $P \rightarrow Y$ is a submersion, so that $\mathcal{X} \times_{\mathcal{Y}} P$ is representable. There exists an open cover $(U_{\alpha})_{\alpha}$ of U and manifolds $(F_{\alpha})_{\alpha}$ such that $U_{\alpha} \times F_{\alpha}$ is open embedded in P and the restriction of $P \rightarrow U$ is the projection $U_{\alpha} \times F_{\alpha} \rightarrow U_{\alpha}$. The fibred products $\mathcal{X} \times_{\mathcal{Y}} U_{\alpha}$ form an open cover of $\mathcal{X} \times_{\mathcal{Y}} U$ (in the stack sense) so that it is enough to show that each $\mathcal{X} \times_{\mathcal{Y}} U_{\alpha}$ is representable. From the abstract properties of fibred products we have

$$(\mathcal{X} \times_{\mathcal{Y}} U_{\alpha}) \times F_{\alpha} = (\mathcal{X} \times_{\mathcal{Y}} U_{\alpha}) \times_{U_{\alpha}} (U_{\alpha} \times F_{\alpha}) = (\mathcal{X} \times_{\mathcal{Y}} P) \times_P (U_{\alpha} \times F_{\alpha}),$$

so that $(\mathcal{X} \times_{\mathcal{Y}} U_{\alpha}) \times F_{\alpha}$ is representable, and we conclude that $\mathcal{X} \times_{\mathcal{Y}} U_{\alpha}$ is representable, as required. \square

It is clear that all representable morphisms of differentiable stacks are weakly representable. Note that any map $F : X \rightarrow Y$ between manifolds is weakly representable, but it is representable if and only if it is a submersion. In fact, a morphism $\mathcal{X} \rightarrow Y$ into a manifold Y is weakly representable if and only if \mathcal{X} is representable. (Note that the notions of representable and weakly representable coincide in the topological setting, see e.g. [23, Lemma 2.6(3)]).

Proposition 2.8 *Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a weakly representable morphism of differentiable stacks. Then the functor F is faithful.*

PROOF: It is well known that F is faithful if (and only if) the functor $F_U : \mathcal{X}_U \rightarrow \mathcal{Y}_U$ is faithful for any manifold U . Since \mathcal{X}_U is a groupoid, it is enough to show that, for any arrow $a : x \rightarrow x$ in \mathcal{X}_U , if $F(a) = \text{id}_{F(x)}$ then $a = \text{id}_x$. Fix such an arrow a .

Since F is weakly representable, there is an atlas $\pi : Y \rightarrow \mathcal{Y}$ such that $\mathcal{X} \times_{\mathcal{Y}} Y$ is representable. The object $x \in \mathcal{X}_U$ corresponds to a morphism $x : U \rightarrow \mathcal{X}$. The fibred product $U' = U \times_{\mathcal{Y}} Y$ is a manifold, and the projection $U' \rightarrow U$ is a surjective submersion. So we can take an open cover (U_α) of U with local sections $U_\alpha \rightarrow U'$ of $U' \rightarrow U$. For any value of the index α , we get a 2-commutative diagram

$$\begin{array}{ccc} & Y & \\ y_\alpha \nearrow & \downarrow \pi & \\ U_\alpha & \xrightarrow{F(x|_{U_\alpha})} & \mathcal{Y}. \end{array}$$

We will show that $a|_{U_\alpha} = \text{id}_{x|_{U_\alpha}}$, so that $a = \text{id}_x$ (since \mathcal{X} is a stack), as desired. Interpreting y_α as an object of Y over U_α , the 2-isomorphism d_α that makes the above diagram 2-commute is interpreted as an isomorphism $d_\alpha : F(x|_{U_\alpha}) \rightarrow \pi(y_\alpha)$ in \mathcal{Y}_U , so that the triple $(x|_{U_\alpha}, d_\alpha, y_\alpha)$ is an object of $\mathcal{X} \times_{\mathcal{Y}} Y$ over U_α . The pair $(a|_{U_\alpha}, \text{id}_{y_\alpha})$ is a morphism of $(x|_{U_\alpha}, d_\alpha, y_\alpha)$ in itself in the category $(\mathcal{X} \times_{\mathcal{Y}} Y)_{U_\alpha}$; this follows from the hypothesis $F(a) = \text{id}_{F(x)}$, which implies that $F(a|_{U_\alpha}) = \text{id}_{F(x|_{U_\alpha})}$. Since $\mathcal{X} \times_{\mathcal{Y}} Y$ is representable, there is a unique arrow of $(x|_{U_\alpha}, d_\alpha, y_\alpha)$ in itself, namely the identity. It follows that $a|_{U_\alpha} = \text{id}_{x|_{U_\alpha}}$ and the proof is complete. \square

Definition 2.9 *A morphism $F : \mathcal{X} \rightarrow \mathcal{Y}$ of differentiable stacks is called a **submersion** (resp. **immersion**) if there exist atlases $X \rightarrow \mathcal{X}$ and $Y \rightarrow \mathcal{Y}$ such that the induced map of manifolds $Y \times_{\mathcal{Y}} X \rightarrow Y$ is a submersion (resp. **immersion**).*

The notion of immersion will be used only for étale stacks (see Section 2.4).

One may verify that, for manifolds, the above definitions coincide with the usual notions of submersion and immersion. Also, any representable morphism is a submersion. Note, however, that submersions (or immersions) need not be representable morphisms; in fact, most submersions we will deal with are of the form $\mathcal{X} \rightarrow Y$, where Y is a manifold, and, as mentioned, such a map cannot be representable unless \mathcal{X} is.

We will be particularly interested in submersions, and it will be useful to have equivalent characterizations of such morphisms.

Proposition 2.10 *Given a morphism of differentiable stacks $F : \mathcal{X} \rightarrow \mathcal{Y}$, the following are equivalent:*

- (a) F is a submersion.
- (b) There exists an atlas $X \rightarrow \mathcal{X}$ such that the composition $X \rightarrow \mathcal{Y}$ is representable.
- (c) For all representable morphisms $U \rightarrow \mathcal{X}$ and $V \rightarrow \mathcal{Y}$ from manifolds U and V , the induced map of manifolds $V \times_{\mathcal{Y}} U \rightarrow V$ is a submersion.

- (d) For all representable morphisms $U \rightarrow \mathcal{X}$ from a manifold U , the composition $U \rightarrow \mathcal{Y}$ is representable.

Submersions satisfy the following natural properties:

Proposition 2.11 *The following holds:*

- (a) *The composition of submersions is a submersion.*
- (b) *A base change of a submersion is a submersion.*
- (c) *Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ and $F' : \mathcal{Y} \rightarrow \mathcal{Z}$ be morphisms of differentiable stacks. If F and $F'F$ are submersions and F is an epimorphism, then F' is a submersion.*

The proofs of Propositions 2.10 and 2.11 can be found in Appendix A.1.

2.4 Lie groupoids and Hilsum-Skandalis maps

We recall some basic facts about Lie groupoids and their relation with differentiable stacks. See e.g. [6, 34, 35] for more details, and [16] for a geometric viewpoint.

A *groupoid* is a category in which all the morphisms are invertible. Hence a groupoid consists of a set G_0 of objects, a set G of morphisms, and structural maps satisfying suitable compatibility conditions. We denote the source and target maps by s and t , we write i for the inversion map, 1 for the unit map, and m for the multiplication:

$$G \times_{s, G_0, t} G \xrightarrow{m} G \xrightarrow{s, t} G_0 \xrightarrow{1} G \xrightarrow{i} G, \quad (2.2)$$

where

$$G \times_{s, G_0, t} G = \{(g, h) \mid s(g) = t(h)\}.$$

We will use the notation $m(g, h) = g \cdot h = gh$, $1(g_0) = 1_{g_0}$ and $i(g) = g^{-1}$. We may also denote the structural maps by s_G , t_G etc. if we need to be more specific. We denote a groupoid by $G \rightrightarrows G_0$, or simply by G if there is no risk of confusion. In this context, a morphism is just a functor.

A **Lie groupoid** $G \rightrightarrows G_0$ is a groupoid in the category of smooth manifolds, such that source and target maps are submersions (necessarily surjective). It is called **étale** if the source map (or, equivalently, the target map) is a local diffeomorphism, or equivalently, if G and G_0 have the same dimension.

A right **action** of a Lie groupoid G on a manifold X is defined by a pair of maps $a : X \rightarrow G_0$ and

$$X \times_{a, G_0, t} G \longrightarrow X, \quad (x, g) \mapsto x \cdot g = xg$$

such that $a(xg) = s(g)$ and

$$(xg)h = x(gh), \quad x1 = x.$$

We say that G **acts on X along $a : X \rightarrow G_0$** , and the map a is often referred to as the **moment map** of the action. A **G -equivariant map** between manifolds equipped with G -actions is a map that commutes with moment maps and actions.

A right **G -bundle** is a manifold P equipped with a right G -action along $a : P \rightarrow G_0$ and a map

$$r : P \rightarrow S,$$

where S is a manifold, such that the action is **on the fibers** of r , i.e., $r(zg) = r(z)$ for composable $z \in P$ and $g \in G$. A right G -bundle is **principal** if r is a surjective submersion and the induced map

$$P \times_{a, G_0, t} G \longrightarrow P \times_S P, \quad (z, g) \mapsto (z, zg), \quad (2.3)$$

is a diffeomorphism. Similar definitions hold for left actions and left bundles.

For a given right action of a Lie groupoid G on a manifold X along $a : X \rightarrow G_0$, there is an associated differentiable stack, called the **quotient stack** and denoted by $[X/G]$. The objects of $[X/G]$ are principal right G -bundles equipped with a G -equivariant map $P \rightarrow X$, while the morphisms of $[X/G]$ are morphisms of principal bundles (over different bases, in general) that commute with the maps to X . Any G -equivariant map $f : X \rightarrow Y$ naturally induces a morphism $[X/G] \rightarrow [Y/G]$.

Remark 2.12 Note that the induced morphism of quotient stacks $[X/G] \rightarrow [Y/G]$ is a submersion (in the sense of Def. 2.9) if f is.

There is a map of stacks $X \rightarrow [X/G]$, taking a smooth map of manifolds $f : U \rightarrow X$ to

$$P = (af)^*G = U \times_{af, G_0, t} G,$$

which is naturally a principal right G -bundle over U , equipped with the equivariant map $P \rightarrow X$, $(u, g) \mapsto f(u)g$. The map $X \rightarrow [X/G]$ defines an atlas of the quotient stack. A particular case of this construction is when $X = G_0$, equipped with its canonical G -action: $(g_0, g) \mapsto s(g)$. In this case the quotient stack is called the **classifying space** of the groupoid G and it is denoted by BG .

There is a close relation between Lie groupoids and differentiable stacks endowed with an atlas. On the one hand, given a Lie groupoid $G \rightrightarrows G_0$, one considers the associated classifying space BG , which comes with an atlas $G_0 \rightarrow BG$ fitting into the following 2-cartesian square:

$$\begin{array}{ccc} G & \xrightarrow{t} & G_0 \\ s \downarrow & & \downarrow \\ G_0 & \longrightarrow & BG. \end{array}$$

Conversely, given a differentiable stack \mathcal{G} endowed with an atlas $G_0 \rightarrow \mathcal{G}$, we define $G = G_0 \times_{\mathcal{G}} G_0$, which has an induced Lie groupoid structure over G_0 such that BG is canonically isomorphic to \mathcal{G} . The Lie groupoid $G \rightrightarrows G_0$ is called a **presentation** of \mathcal{G} (and of BG). We say that the differentiable stack \mathcal{G} is **étale** if it can be presented by an étale Lie groupoid.

Example 2.13 Given a (right) action of a Lie groupoid $G \rightrightarrows G_0$ on a manifold X along $a : X \rightarrow G_0$, the quotient stack $[X/G]$ is presented by the *action* (or *translation*) groupoid $X \rtimes G$: its space of arrows is $X \times_{a, G_0, t} G$, the source map is $(x, g) \mapsto xg$, the target map is $(x, g) \mapsto x$, and the multiplication is given by $(x, g)(y, h) = (x, gh)$.

A G - H -**bibundle** is defined by a manifold P and Lie groupoids $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ so that P carries a left G -action along $a : P \rightarrow G_0$ and on the fibers of b , and a right H -action along $b : P \rightarrow H_0$ and on the fibers of a , in such a way that the two actions commute. We represent such bibundle by the diagram

$$\begin{array}{ccc} G & & H \\ \Downarrow & & \Downarrow \\ G_0 & \xleftarrow{a} P \xrightarrow{b} & H_0. \end{array} \quad (2.4)$$

We will simply write P for the bibundle. Two G - H bibundles P and Q are **isomorphic** if there is a diffeomorphism $P \rightarrow Q$ preserving the actions.

A bibundle (2.4) is called **right principal** if it is a principal H -bundle over G_0 (in particular, a is a surjective submersion), and it is called **biprincipal** if it is also a principal G -bundle over H_0 (so that b is also a surjective submersion). We say that two groupoids G and H are **Morita equivalent** if there exists a biprincipal G - H -bibundle.

Given a right principal H - K -bibundle Q ,

$$\begin{array}{ccc} H & & K \\ \Downarrow & & \Downarrow \\ H_0 & \longleftarrow Q \longrightarrow & K_0, \end{array}$$

and a right principal H -bundle P , the quotient of $P \times_{H_0} Q$ by the H -action $(z, w)h = (zh, h^{-1}w)$ is a right principal K -bundle, denoted by $P \otimes_H Q$. Moreover, if P is a right principal G - H -bibundle, so that we have

$$\begin{array}{ccccc} G & & H & & K \\ \Downarrow & & \Downarrow & & \Downarrow \\ G_0 & \longleftarrow P \longrightarrow & H_0 & \longleftarrow Q \longrightarrow & K_0, \end{array}$$

then $P \otimes_H Q$ is naturally a right principal G - K -bibundle,

$$\begin{array}{ccc} G & & K \\ \Downarrow & & \Downarrow \\ G_0 & \longleftarrow P \otimes_H Q \longrightarrow & K_0. \end{array}$$

If P and Q are biprincipal, then so is $P \otimes_H Q$, which implies that Morita equivalence of Lie groupoids is a transitive relation (the fact that it is symmetric and reflexive is straightforward).

A **Hilsum-Skandalis map** between Lie groupoids G and H is an isomorphism class of right principal G - H -bibundles. Any groupoid morphism $\phi : G \rightarrow H$ gives rise to a Hilsum-Skandalis map through the right principal G - H -bibundle defined by the fibred product

$$G_0 \times_{\phi_0, H_0, t} H, \quad (2.5)$$

with actions given by $g(g_0, h) = (t(g), \phi(g)h)$ and $(g_0, h)h' = (g_0, hh')$. Since right principal bibundles represent Hilsum-Skandalis maps, we also refer to them as **Hilsum-Skandalis bi-bundles**.

Given a right principal G - H -bibundle Q , there is an associated morphism between the stacks BG and BH , sending a principal right G -bundle P (that is, an object of BG) to the principal right H -bundle $P \otimes_G Q$. In this way a Hilsum-Skandalis map from G to H induces a morphism $BG \rightarrow BH$, defined up to 2-isomorphism. In particular, any groupoid morphism $\phi : G \rightarrow H$ gives rise to a morphism $BG \rightarrow BH$ via (2.5); moreover, this map $BG \rightarrow BH$ is an isomorphism whenever the groupoid morphism ϕ is a **weak equivalence**, that is, when the following two conditions are satisfied (see e.g. [34, Sec. 5.4]): (1) the map $s \circ \text{pr}_2 : G_0 \times_{\phi_0, H_0, t} H \rightarrow H_0$ is a surjective submersion, and (2) the square

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \downarrow (s,t) & & \downarrow (s,t) \\ G_0 \times G_0 & \xrightarrow{\phi_0 \times \phi_0} & H_0 \times H_0 \end{array}$$

is cartesian.

Conversely, any morphism $BG \rightarrow BH$ of stacks, taken up to 2-isomorphism, is presented by a unique Hilsum-Skandalis map between the groupoids G and H . Indeed, for a given morphism $F : BG \rightarrow BH$, a representative of the corresponding Hilsum-Skandalis map is given by the bibundle

$$G_0 \times_{BH} H_0,$$

which can be identified with $F(G)$ as a principal right H -bundle over G_0 (here we view G as a principal G -bundle relative to right multiplication); we denote by $r : F(G) \rightarrow G_0$ the projection. The left G -action on $F(G)$ is given by $F(m) : G \times_{s, G_0, r} F(G) = s^*(F(G)) \rightarrow F(G)$, where we view the multiplication $m : G \times_{s, G_0, t} G = s^*G \rightarrow G$ as a morphism of principal right G -bundles (covering the map $t : G \rightarrow G_0$).

Given Lie groupoids X , G , and H , and morphisms $a : X \rightarrow H$, $b : G \rightarrow H$ so that either $a_0 : X_0 \rightarrow H_0$ or $b_0 : G_0 \rightarrow H_0$ is a submersion, we denote by $X \times_H G$ the Lie groupoid given by their **weak fibred product** [34, Sec. 5.3]: its space of objects is $X_0 \times_{a_0, H_0, s} H \times_{t, H_0, b_0} G_0$, its space of arrows is $X \times_{sa, H_0, s} H \times_{t, H_0, sb} G$, and multiplication is given by $(x, h, g)(x', h', g') = (xx', h', gg')$ (source and target maps are given by $(x, h, g) \mapsto (s(x), h, s(g))$ and $(x, h, g) \mapsto (t(x), b(g)ha(x)^{-1}, t(g))$, respectively). Note that there is a canonical map

$$B(X \times_H G) \rightarrow BX \times_{BH} BG, \quad (2.6)$$

induced by the natural maps $B(X \times_H G) \rightarrow BX$ and $B(X \times_H G) \rightarrow BG$, which correspond to the groupoid morphisms $X \times_H G \rightarrow X$ and $X \times_H G \rightarrow G$. We recall the following fact:

Proposition 2.14 *The canonical map $B(X \times_H G) \rightarrow BX \times_{BH} BG$ in (2.6) is an isomorphism.*

PROOF: We will describe the inverse map. Given (P, φ, Q) in $BX \times_{BH} BG$ over a manifold U , we have that $P \times_U Q$ is a principal right $X \times_H G$ bundle over U as follows. Let $j : P \rightarrow X_0$ and $k : Q \rightarrow G_0$ be the moment maps for the actions on P and Q , and note that the images of P and Q in BH are the H -bundles $(P \times_{a_0 j, H_0, t} H)/X$ and $(Q \times_{b_0 k, H_0, t} H)/G$, respectively. Denoting an object in $X \times_H G$ by $(x_0, h, g_0) \in X_0 \times_{H_0} H \times_{H_0} G_0$, the moment map for the action on $P \times_U Q$ is the map

$$(p, q) \mapsto (x_0, h, g_0),$$

where $x_0 = j(p)$, $g_0 = k(q)$, and $h \in H$ is defined as follows: given $(z, z') \in P \times_U Q$ and the H -equivariant map $\varphi : (P \times_{a_0 j, H_0, t} H)/X \rightarrow (Q \times_{b_0 k, H_0, t} H)/G$, we let $[z, 1]$ denote the X -orbit of $(z, 1) \in P \times_{H_0} H$, and h is uniquely defined by the condition that $[z', h] = \varphi([z, 1])$. The right action of $X \times_H G$ on $P \times_U Q$ is $(z, z')(x, h, g) = (zx, z'g)$. \square

Some examples of Hilsum-Skandalis bibundles

The next two lemmas give explicit examples of Hilsum-Skandalis bibundles that we will need later, see Sections 5.2 and 5.4.

Lemma 2.15 *Let \mathcal{G} and \mathcal{X} be differentiable stacks, presented by $G \rightrightarrows G_0$ and $X \rightrightarrows X_0$, and let M be a manifold. Consider a submersion $\mathcal{G} \rightarrow M$ and a morphism $\mathcal{X} \rightarrow M$. Then the fibred product $\mathcal{X} \times_M \mathcal{G}$ is a differentiable stack presented by $X \times_M G \rightrightarrows X_0 \times_M G_0$, and the projection $\mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X}$ corresponds to the Hilsum-Skandalis bibundle $G_0 \times_M X$, with actions given by $(x, g)(g_0, x') = (t(g), xx')$ (for the moment map $(g_0, x) \mapsto (t(x), g_0)$) and $(g_0, x')x = (g_0, x'x)$ (for the moment map $(g_0, x) \mapsto s(x)$).*

PROOF: The assertion that $\mathcal{X} \times_M \mathcal{G} = BX \times_M BG$ is a differentiable stack presented by $X \times_M G \rightrightarrows X_0 \times_M G_0$ is a consequence of the canonical isomorphism $B(X \times_M G) \rightarrow BX \times_M BG$ of Prop. 2.14 (the fact that $BG \rightarrow M$ is a submersion guarantees that $G_0 \rightarrow M$ is a submersion, and hence $X \times_M G$ is a Lie groupoid). We note that this isomorphism commutes with the projections to BX , where the projection $B(X \times_M G) \rightarrow BX$ is associated with the groupoid projection $X \times_M G \rightarrow X$. We conclude (cf. (2.5)) that this projection is represented by the Hilsum-Skandalis bibundle $(X_0 \times_M G_0) \times_{X_0} X \simeq G_0 \times_M X$, with the actions described in the statement of the Lemma. \square

Lemma 2.16 *Let E_1 and E_2 be Hilsum-Skandalis bibundles for stack morphisms $F_1 : BX \rightarrow BY$ and $F_2 : BX \rightarrow BZ$, respectively. Assume that one of the two maps is a submersion. Let M be a manifold, and suppose that we have maps $F_3 : BY \rightarrow M$ and $F_4 : BZ \rightarrow M$ so that $F_3 F_1 = F_4 F_2$. Then $E_1 \times_{X_0} E_2$ is a Hilsum-Skandalis bibundle for the induced morphism $F : BX \rightarrow BY \times_M BZ$.*

PROOF: Using the identifications $E_1 = F_1(X)$ and $E_2 = F_2(X)$ and denoting by $\psi : BY \times_M BZ \rightarrow B(Y \times_M Z)$ the isomorphism described in the proof of Prop. 2.14, we conclude that $\psi F(X) = F_1(X) \times_{X_0} F_2(X)$. The natural left X -action on $F_1(X) \times_{X_0} F_2(X)$ coincides with the diagonal action. \square

3 Stacky Lie groupoids and actions

In this section we define stacky Lie groupoids and their actions on differentiable stacks.

3.1 Stacky Lie groupoids

Let M be a manifold³ and \mathcal{G} be a category fibred in groupoids over \mathcal{C} . The manifold M is to be thought of as the space of units of a groupoid structure on \mathcal{G} . We use the same symbol M for the differentiable stack associated with M . We will consider the following data and conditions:

(g1) Morphisms s , t , 1 , i , and m (called *source*, *target*, *unit*, *inverse*, and *multiplication* maps, respectively) as follows:

$$\begin{aligned} \mathcal{G} &\xrightarrow{s,t} M \xrightarrow{1} \mathcal{G} \xrightarrow{i} \mathcal{G}, \\ \mathcal{G} \times_{s,M,t} \mathcal{G} &\xrightarrow{m} \mathcal{G}, \quad m(g, h) = g \cdot h = gh. \end{aligned}$$

For the identity and the inverse we will also use the notation

$$1(x) = 1_x = 1, \quad i(g) = g^{-1}.$$

The multiplication morphism is defined on the fibred product of $s : \mathcal{G} \rightarrow M$ and $t : \mathcal{G} \rightarrow M$,

$$\begin{array}{ccc} \mathcal{G} \times_{s,M,t} \mathcal{G} & \xrightarrow{n_s} & \mathcal{G} \\ n_t \downarrow & & \downarrow t \\ \mathcal{G} & \xrightarrow{s} & M, \end{array} \tag{3.1}$$

³The manifold M is assumed to be Hausdorff and second countable.

which generalizes the space of composable arrows on Lie groupoids.

(g2) The morphisms s , t , 1 , i , and m are assumed to satisfy the following identities

$$\begin{aligned} s1 &= \text{id}_M \\ t1 &= \text{id}_M \\ si &= t \\ ti &= s \\ sm &= sn_s \\ tm &= tn_t. \end{aligned}$$

We also require axioms analogous to those of Lie groupoids, but now in a weaker sense (see (g3) below). Let us consider the morphisms

- $m(\text{id} \times m) : \mathcal{G} \times_{s,M,t} \mathcal{G} \times_{s,M,t} \mathcal{G} \longrightarrow \mathcal{G}$ and $m(m \times \text{id}) : \mathcal{G} \times_{s,M,t} \mathcal{G} \times_{s,M,t} \mathcal{G} \longrightarrow \mathcal{G}$, encoding the two possible ways to compose three elements in \mathcal{G} ;
- $m\langle 1t, \text{id} \rangle : \mathcal{G} \longrightarrow \mathcal{G}$ and $m\langle \text{id}, 1s \rangle : \mathcal{G} \longrightarrow \mathcal{G}$, encoding multiplication by the identity on the left and on the right. Here we use the notation $\langle 1t, \text{id} \rangle : \mathcal{G} \rightarrow \mathcal{G} \times_{s,M,t} \mathcal{G}$ for the map induced by $1t : \mathcal{G} \rightarrow \mathcal{G}$ and $\text{id} : \mathcal{G} \rightarrow \mathcal{G}$.
- $m\langle i, \text{id} \rangle : \mathcal{G} \longrightarrow \mathcal{G}$ and $m\langle \text{id}, i \rangle : \mathcal{G} \longrightarrow \mathcal{G}$, encoding multiplication by the inverse on the left and on the right.

(g3) Five 2-isomorphisms α , λ , ρ , ι_l and ι_r ,

$$\begin{aligned} \alpha &: m(\text{id} \times m) \xrightarrow{\sim} m(m \times \text{id}), \\ \lambda &: m\langle 1t, \text{id} \rangle \xrightarrow{\sim} \text{id}, \quad \rho : m\langle \text{id}, 1s \rangle \xrightarrow{\sim} \text{id}, \\ \iota_l &: m\langle i, \text{id} \rangle \xrightarrow{\sim} 1s, \quad \iota_r : m\langle \text{id}, i \rangle \xrightarrow{\sim} 1t. \end{aligned}$$

These 2-isomorphisms represent weaker forms of the associativity, identity and inversion axioms on groupoids, respectively.

(g4) The 2-isomorphisms α , λ , ρ , ι_l and ι_r satisfy the *higher coherence* conditions given by the commutativity of the following diagrams, displayed with their corresponding labels on the left⁴:

$$\begin{array}{ccc} (kghl) : & \begin{array}{ccccc} (kg \cdot h)l & \xleftarrow{\alpha} & kg \cdot hl & \xleftarrow{\alpha} & k(g \cdot hl) \\ \alpha \cdot \text{id} \uparrow & & & & \downarrow \text{id} \cdot \alpha \\ (k \cdot gh)l & \xleftarrow{\alpha} & & & k(gh \cdot l) \end{array} \\ (1gh) : & \begin{array}{ccc} 1 \cdot gh & \xrightarrow{\alpha} & 1g \cdot h \\ \lambda \downarrow & \swarrow \lambda \cdot \text{id} & \\ gh & & \end{array} & (g1h) : \begin{array}{ccc} g \cdot 1h & \xrightarrow{\alpha} & g1 \cdot h \\ \text{id} \cdot \lambda \downarrow & \swarrow \rho \cdot \text{id} & \\ gh & & \end{array} \end{array}$$

⁴In order to simplify our notation, we will often write expressions of the form $(gh)l$ simply as $gh \cdot l$; in other words, we will implicitly assume the priority of juxtaposition over “.”.

$$\begin{array}{ccc}
(gh1) : & g \cdot h1 \xrightarrow{\alpha} gh \cdot 1 & (gg^{-1}g) : \\
& \text{id} \cdot \rho \downarrow \swarrow \rho & g1 \xrightarrow{\rho} g \xleftarrow{\lambda} 1g \\
& gh & \text{id} \cdot \iota_l \uparrow \quad \quad \quad \uparrow \iota_r \cdot \text{id} \\
& & g(g^{-1} \cdot g) \xrightarrow{\alpha} (g \cdot g^{-1})g
\end{array}$$

where $k, g, h, l \in \mathcal{G}$ are such that the compositions make sense and the 1's are the appropriate identities of \mathcal{G} .

Definition 3.1 A **groupoid** in $\text{CFG}_{\mathcal{C}}$ is defined by a manifold M , an object \mathcal{G} in $\text{CFG}_{\mathcal{C}}$, and morphisms $s, t, 1, i, m$ and 2-isomorphisms $\alpha, \lambda, \rho, \iota_l, \iota_r$ as in (g1), (g2), (g3), (g4) above.

Note that the first two identities in (g2) imply that s and t are *epimorphisms*.

Remark 3.2 Since M is a manifold (so, as a category fibred in groupoids, it is fibred in sets), any two isomorphic morphisms into M must coincide. It follows that (3.1), which is in principle a 2-fibred product, is a 1-fibred product; in particular, (3.1) commutes in the strict sense. For the same reason, we require equalities (rather than isomorphisms) in the identities in (g2).

Remark 3.3 For the higher coherences (g4), we selected a set of conditions that we explicitly use throughout the paper and that generates other coherences, but is not meant to be minimal (i.e., it may contain redundancies). For more on higher coherences, see [24, 25].

A groupoid in $\text{CFG}_{\mathcal{C}}$ will be alternatively called a **cfg-groupoid**; we use the notation $\mathcal{G} \rightrightarrows M$, or simply \mathcal{G} . A **cfg-group** is a cfg-groupoid for which the base manifold M is a point.

Remark 3.4 Given a cfg-groupoid $\mathcal{G} \rightrightarrows M$ and a smooth map $\iota : N \rightarrow M$, we consider the fibred product

$$\begin{array}{ccc}
\mathcal{G}_N & \longrightarrow & N \times N \\
\downarrow & & \downarrow (\iota, \iota) \\
\mathcal{G} & \xrightarrow[t, s]{} & M \times M.
\end{array}$$

One may verify that $\mathcal{G}_N \rightrightarrows N$ is naturally a cfg-groupoid, with operations and higher coherences inherited from those for \mathcal{G} .

As a particular instance of Remark 3.4, we consider, for each $x \in M$, the fibred product

$$\begin{array}{ccc}
\mathcal{G}_x & \longrightarrow & \{x\} \\
\downarrow & & \downarrow \text{diag} \\
\mathcal{G} & \xrightarrow[t, s]{} & M \times M,
\end{array} \tag{3.2}$$

which is a cfg-group, called the **isotropy** group of \mathcal{G} at x .

One can also consider s -fibres, and analogously t -fibres, defined for each $x \in M$ as the category fibred in groupoids resulting from the fibred product

$$\begin{array}{ccc}
s^{-1}(x) & \longrightarrow & \{x\} \\
\downarrow & & \downarrow \\
\mathcal{G} & \xrightarrow{s} & M.
\end{array} \tag{3.3}$$

Definition 3.5 A *stacky groupoid* is a groupoid $\mathcal{G} \rightrightarrows M$ in $\text{CFG}_{\mathcal{C}}$ such that \mathcal{G} is a stack. A *stacky Lie groupoid* is a groupoid $\mathcal{G} \rightrightarrows M$ in $\text{CFG}_{\mathcal{C}}$ such that \mathcal{G} is a differentiable stack, and source and target (epi)morphisms s, t are submersions.

A stacky Lie groupoid $\mathcal{G} \rightrightarrows M$ is called *étale* if \mathcal{G} is an étale differentiable stack.

We have some important classes of examples.

Example 3.6 A particular class of stacky Lie groups is given by the (strict) *2-groups*, defined as Lie groupoids $G \rightrightarrows G_0$ where both G and G_0 are Lie groups, and so that the Lie-group multiplication and inversion maps define morphisms of Lie groupoids,

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ \Downarrow & & \Downarrow \\ G_0 \times G_0 & \longrightarrow & G_0, \end{array} \quad \begin{array}{ccc} G & \longrightarrow & G \\ \Downarrow & & \Downarrow \\ G_0 & \longrightarrow & G_0. \end{array}$$

In this case $BG = [G_0/G]$ inherits the structure of a stacky Lie group:

$$m : BG \times BG \rightarrow BG, \quad i : BG \rightarrow BG.$$

As an example, consider a homomorphism of abelian Lie groups $\varphi : A \rightarrow K$, and the action of A on K by $k \mapsto k + \varphi(a)$. Then the action groupoid $A \ltimes K \rightrightarrows K$ defines a 2-group, where the additional group structure on $A \times K$ is the direct product. This is a special instance of the well-known fact that 2-groups admit an equivalent description as *crossed modules* [9] (see also [21, 38]). In particular, taking $K = \{e\}$, we see that BA is a 2-group.

Remark 3.7 As shown in [48], every *étale* stacky Lie group (connected, finite dimensional) can be strictified, i.e., it is isomorphic to a 2-group as in the previous example (arising from a crossed module). However, this is no longer true without the étaleness condition: an example is given by the *string Lie 2-group*, which is a (non-étale) stacky Lie group obtained by a central extension [44] of a simply connected Lie group G by BS^1 ; this stacky Lie group cannot be strictified by finite-dimensional models, see [2]. The reader can find more on stacky Lie groups e.g. in [7] (see also [2, 22], and [55] for infinite dimensional examples arising from central extensions).

Example 3.8 Extending the previous example, one may consider (strict) 2-groupoids; these are defined as *double Lie groupoids* [10, 30] of the form

$$\begin{array}{ccc} G & \rightrightarrows & M \\ \Downarrow & \text{id} \Downarrow \text{id} & \\ G_0 & \rightrightarrows & M, \end{array}$$

where the vertical groupoid on the right is the trivial groupoid; in this case, similarly to the previous example, BG inherits the structure of a stacky Lie groupoid over M (note that the source and target maps $BG \rightrightarrows M$ are submersions, as a consequence of Prop. 2.10(b)). For a description of 2-groupoids in terms of crossed modules, see e.g. [10, 21].

Example 3.9 Étale stacky Lie groupoids naturally arise as global objects associated with *Lie algebroids*. Given a Lie algebroid $A \rightarrow M$, following [49], the path-space construction of [15, 45] considers the Banach manifold of A -paths along with the (finite-codimensional) foliation

determined by A -homotopies. This leads to two natural étale stacky Lie groupoids associated with A , depending on whether one models the leaf space of A -paths by A -homotopies using the monodromy groupoid or the holonomy groupoid of the foliation. These stacky Lie groupoids are denoted by

$$\mathcal{G}(A) \rightrightarrows M, \quad \text{and} \quad \mathcal{H}(A) \rightrightarrows M,$$

respectively.

Conversely, it is also shown in [49] that an étale stacky Lie groupoid $\mathcal{G} \rightrightarrows M$ defines a local Lie groupoid $G_{loc} \rightrightarrows M$, hence a Lie algebroid over M .

Remark 3.10 Given cfg-groupoids $\mathcal{G}_i \rightrightarrows M_i$ for $i = 1, 2$, there is a natural cfg-groupoid $\mathcal{G}_1 \times \mathcal{G}_2 \rightrightarrows M_1 \times M_2$ that we call the *product groupoid* of the original groupoids. If each \mathcal{G}_i is a stacky Lie groupoid then so is the product groupoid.

Additional properties of stacky Lie groupoids

For a stacky Lie groupoid $\mathcal{G} \rightrightarrows M$, the \mathbf{s} -fibres (3.3) (resp. \mathbf{t} -fibres) are differentiable stacks. Indeed, given an atlas $\pi : G_0 \rightarrow \mathcal{G}$, the maps

$$\mathbf{s}_0 = \mathbf{s} \circ \pi : G_0 \rightarrow M, \quad \mathbf{t}_0 = \mathbf{t} \circ \pi : G_0 \rightarrow M, \quad (3.4)$$

are submersions, so $\mathbf{s}_0^{-1}(x)$ (resp. $\mathbf{t}_0^{-1}(x)$) is a smooth manifold for all $x \in M$, which is naturally an atlas for $\mathbf{s}^{-1}(x)$ (resp. $\mathbf{t}^{-1}(x)$).

Lemma 3.11 *Let $\mathcal{G} \rightrightarrows M$ be an étale stacky Lie groupoid, and let $\pi : G_0 \rightarrow \mathcal{G}$ be an étale atlas. Then for any $x \in M$, the restriction $\mathbf{t}_0|_{\mathbf{s}_0^{-1}(x)} : \mathbf{s}_0^{-1}(x) \rightarrow M$ is a constant-rank map.*

PROOF: The unit map $1 : M \rightarrow \mathcal{G}$ gives rise (by a choice of local section of the corresponding Hilsum-Skandalis bibundle) to a map $\epsilon_0 : V_x \rightarrow G_0$, where V_x is an open neighborhood of x , such that $\mathbf{s}_0 \circ \epsilon_0 = \text{id}_M$ and $\mathbf{t}_0 \circ \epsilon_0 = \text{id}_M$; in particular, ϵ_0 is an immersion, so we can assume it is an embedding for V_x small enough.

Let $y = \epsilon_0(x) \in G_0$ and pick any $z \in \mathbf{s}_0^{-1}(x)$, so that $\mathbf{s}_0(y) = \mathbf{t}_0(y) = x = \mathbf{s}_0(z)$. Let $N = \mathbf{s}_0^{-1}(x)$. We will show that $d\mathbf{t}_0|_{T_y N}$ and $d\mathbf{t}_0|_{T_z N}$ have the same rank. The multiplication \mathbf{m} defines a Hilsum-Skandalis bibundle, and a choice of local section σ around $(z, y) \in G_0 \times_{\mathbf{s}_0, M, \mathbf{t}_0} G_0$ leads to a map $m_0 : U_z \times_{\mathbf{s}_0, M, \mathbf{t}_0} U_y \rightarrow G_0$, where U_z and U_y are open neighborhoods of z and y in G_0 , respectively. Moreover, this local section can be chosen in such a way that m_0 satisfies

$$m_0(z', \epsilon_0(\mathbf{s}_0(z'))) = z', \quad \text{for all } z' \in U_z. \quad (3.5)$$

For that, one uses the higher morphism ρ in **(g3)**, along with the same arguments as in [49, Thm. 5.2]. Additionally,

$$m_0(z, \cdot) : \mathbf{t}_0^{-1}(x) \cap U_y \rightarrow \mathbf{t}_0^{-1}(\mathbf{t}_0(z)) \quad (3.6)$$

is a local diffeomorphism around y . This follows from the observation that, for any object g in \mathcal{G} , left-multiplication by g gives rise to a morphism of stacks $L_g : \mathbf{t}^{-1}(\mathbf{s}(g)) \rightarrow \mathbf{t}^{-1}(\mathbf{t}(g))$, which is an isomorphism. Hence, for $g = \pi(z)$, L_g defines a Morita bibundle $\mathbf{t}_0^{-1}(\mathbf{s}_0(z)) \xleftarrow{a} E_{L_g} \xrightarrow{b} \mathbf{t}_0^{-1}(\mathbf{t}_0(z))$ between the corresponding groupoid presentations. The local section σ induces a local section σ' of the moment map a around y , and the resulting map $b \circ \sigma'$, which is a local diffeomorphism (since the Morita bibundle E_{L_g} is étale), agrees with $m_0(z, \cdot)$.

Let us choose a submanifold $\Sigma \subset U_z$, with $z \in \Sigma$, with the property that $s_0|_\Sigma$ and $t_0|_\Sigma$ are open embeddings. By possibly restricting the open neighborhoods, we can assume that $s_0(U_z) = s_0(\Sigma) = s_0(U_y) = V_x$, and $\epsilon_0 : V_x \rightarrow G_0$ is an embedding. Consider the neighborhood of $t_0(z)$ in M given by $V_{t_0(z)} = t_0(U_z)$. Then we have a diffeomorphism $f_\Sigma : V_x \rightarrow V_{t_0(z)}$ defined by $f_\Sigma(x') = t_0(z')$, where $z' \in \Sigma$ is unique so that $s_0(z') = x'$. We also have a map $F_\Sigma : t_0^{-1}(V_x) \cap U_y \rightarrow t_0^{-1}(V_{t_0(z)})$ given by “left multiplication”: $F_\Sigma(y') = m_0(z', y')$, where $z' \in \Sigma$ is unique such that $s_0(z') = t_0(y')$. This map is such that $F_\Sigma(y) = z$, and it preserves s_0 -fibres: $s_0 \circ F_\Sigma = s_0$. Note also that $f_\Sigma \circ t_0 = t_0 \circ F_\Sigma$, so in order to conclude that the ranks of $dt_0|_{T_y N}$ and $dt_0|_{T_z N}$ agree, it suffices to show that F_Σ is a local diffeomorphism around y . The tangent spaces at y (resp. z) to the submanifolds $\epsilon_0(V_x)$ and $t_0^{-1}(x)$ (resp. Σ and $t_0^{-1}(t_0(z))$) are such that their direct sum is $T_y G_0$ (resp. $T_z G_0$), so the result follows from the observation that, around y , both $F_\Sigma|_{\epsilon_0(V_x)} : \epsilon_0(V_x) \rightarrow \Sigma$ and $F_\Sigma|_{t_0^{-1}(x) \cap U_y} = m_0(z, \cdot) : t_0^{-1}(x) \cap U_y \rightarrow t_0^{-1}(t_0(z))$ are local diffeomorphisms (cf. (3.5) and (3.6)). \square

Corollary 3.12 *For $\mathcal{G} \rightrightarrows M$ a stacky Lie groupoid with étale atlas $\pi : G_0 \rightarrow \mathcal{G}$, if the restriction $t_0|_{s_0^{-1}(x)} : s_0^{-1}(x) \rightarrow M$ is onto, then it is a submersion.*

We will refer to a stacky Lie groupoid admitting an étale atlas as in Corollary 3.12 as **transitive**.

Remark 3.13 More generally, the map in the previous corollary is a submersion onto leaves of the underlying Lie algebroid.

We can now show two key properties of étale stacky Lie groupoids.

Proposition 3.14 *Let $\mathcal{G} \rightrightarrows M$ be an étale stacky Lie groupoid. The following holds:*

- (i) *The unit map $1 : M \rightarrow \mathcal{G}$ is an immersion.*
- (ii) *\mathcal{G}_x is an (étale) stacky Lie group for all $x \in M$,*

PROOF:

For (i), it follows directly from Def. 2.9 that it suffices to show that, for an atlas $\pi : G_0 \rightarrow \mathcal{G}$, the induced map $f : M \times_{1, \mathcal{G}, \pi} G_0 \rightarrow G_0$ is an immersion. Consider the induced map $\pi_M : M \times_{1, \mathcal{G}, \pi} G_0 \rightarrow M$, which is a surjective submersion. Taking any local section σ of π_M and using that $s \circ 1 = \text{id}_M$ and $\pi \circ f \cong 1 \circ \pi_M$, it follows that

$$s \circ \pi(f(\sigma(x))) = x, \quad (3.7)$$

for all x in the domain of σ . The condition that \mathcal{G} is étale implies that we can take π to be étale, hence π_M is étale, and σ is a local diffeomorphism. It then follows from (3.7) that f is an immersion.

For (ii), note that by considering an étale atlas $\pi : G_0 \rightarrow \mathcal{G}$ in (3.2), the natural map $s_0^{-1}(x) \cap t_0^{-1}(x) \rightarrow \mathcal{G}_x$ defines an atlas as long as $s_0^{-1}(x) \cap t_0^{-1}(x)$ is a manifold, which is a direct consequence of Lemma 3.11. \square

3.2 Actions

Consider $\mathcal{G} \rightrightarrows M$, a groupoid in $\mathbf{CFG}_{\mathcal{C}}$, as in Definition 3.1, and let $\mathcal{X} \in \mathbf{Obj}(\mathbf{CFG}_{\mathcal{C}})$. The definition of a (right) action of \mathcal{G} on \mathcal{X} requires the following data and conditions:

(a1) Morphisms \mathbf{a} (the *moment map*) and \mathbf{A} (the *action map*),

$$\begin{aligned} \mathcal{X} &\xrightarrow{\mathbf{a}} M, \\ \mathcal{X} \times_{\mathbf{a}, M, \mathbf{t}} \mathcal{G} &\xrightarrow{\mathbf{A}} \mathcal{X}, \quad \mathbf{A}(x, g) = x \cdot g = xg, \end{aligned}$$

where, in the definition of \mathbf{A} , we use the fibred product of $\mathbf{a} : \mathcal{X} \rightarrow M$ and $\mathbf{t} : \mathcal{G} \rightarrow M$,

$$\begin{array}{ccc} \mathcal{X} \times_{\mathbf{a}, M, \mathbf{t}} \mathcal{G} & \xrightarrow{n_{\mathbf{a}}} & \mathcal{G} \\ \downarrow & & \downarrow \mathbf{t} \\ \mathcal{X} & \xrightarrow{\mathbf{a}} & M. \end{array}$$

(a2) The morphisms \mathbf{a} and \mathbf{A} are assumed to satisfy the following identity:

$$\mathbf{a}\mathbf{A} = sn_{\mathbf{a}}.$$

We also require additional action axioms in weak form. Let us consider the morphisms

- $\mathbf{A}(\mathrm{id} \times \mathbf{m}) : \mathcal{X}_{\mathbf{a} \times \mathbf{t}} \times_M \mathcal{G}_{\mathbf{s} \times \mathbf{t}} \rightarrow \mathcal{X}$ and $\mathbf{A}(\mathbf{A} \times \mathrm{id}) : \mathcal{X}_{\mathbf{a} \times \mathbf{t}} \times_M \mathcal{G}_{\mathbf{s} \times \mathbf{t}} \rightarrow \mathcal{X}$, encoding the two ways in which one can act on \mathcal{X} by two elements of \mathcal{G} ,
- $\mathbf{A}(\mathrm{id}, 1_{\mathbf{a}}) : \mathcal{X} \rightarrow \mathcal{X}$, encoding multiplication by the identity.

(a3) Two 2-isomorphisms

$$\begin{aligned} \beta : \mathbf{A}(\mathrm{id} \times \mathbf{m}) &\xrightarrow{\sim} \mathbf{A}(\mathbf{A} \times \mathrm{id}), \\ \varepsilon : \mathbf{A}(\mathrm{id}, 1_{\mathbf{a}}) &\xrightarrow{\sim} \mathrm{id}. \end{aligned}$$

These 2-isomorphisms represent weaker versions of the associativity and multiplication by identity axioms for groupoid actions.

(a4) The 2-isomorphisms β and ε satisfy the *higher coherence* conditions given by the commutativity of the following diagrams, displayed with their corresponding labels on the left (we follow the notation explained in (g4)):

$$\begin{array}{ccc} (xghl) : & \begin{array}{ccc} x(g \cdot hl) & \xrightarrow{\beta} & xg \cdot hl \xrightarrow{\beta} (xg \cdot h)l \\ \mathrm{id} \cdot \alpha \downarrow & & \uparrow \beta \cdot \mathrm{id} \\ x(gh \cdot l) & \xrightarrow{\beta} & (x \cdot gh)l \end{array} \\ (x1g) : & \begin{array}{ccc} x \cdot 1g & \xrightarrow{\beta} & x1 \cdot g \\ \mathrm{id} \cdot \lambda \downarrow & \swarrow \varepsilon \cdot \mathrm{id} & \\ xg & & \end{array} \\ (xg1) : & \begin{array}{ccc} x \cdot g1 & \xrightarrow{\beta} & xg \cdot 1 \\ \mathrm{id} \cdot \rho \downarrow & \swarrow \varepsilon & \\ xg & & \end{array} \end{array}$$

where $x \in \mathcal{X}$ and $g, h, l \in \mathcal{G}$ are such that the compositions in the diagrams make sense and the 1's are the appropriate identities of \mathcal{G} .

Definition 3.15 Let $\mathcal{G} \rightrightarrows M$ be a groupoid in $\text{CFG}_{\mathcal{C}}$ and $\mathcal{X} \in \text{Obj}(\text{CFG}_{\mathcal{C}})$. A **right action** of \mathcal{G} on \mathcal{X} is defined by morphisms \mathbf{a}, \mathbf{A} and 2-isomorphisms β, ε as in **(a1)**, **(a2)**, **(a3)**, **(a4)**.

Regarding **(a4)**, a similar observation to Remark 3.3 applies. We will say that \mathcal{G} **acts on \mathcal{X} along \mathbf{a}** , or alternatively that \mathcal{G} acts on $\mathbf{a} : \mathcal{X} \rightarrow M$, to make the moment map explicit. In this paper, we will be mostly interested in actions of stacky Lie groupoids on differentiable stacks.

Given an action, we define its associated **action-projection map** as

$$\Delta = (\text{pr}_1, \mathbf{A}) : \mathcal{X} \times_{\mathbf{a}, M, \mathbf{t}} \mathcal{G} \longrightarrow \mathcal{X} \times \mathcal{X}. \quad (3.8)$$

We will often denote this map by indicating how the functor acts on objects: $(x, g) \mapsto (x, gx)$.

Remark 3.16 (Left actions) A left action of $\mathcal{G} \rightrightarrows M$ on $\mathbf{a} : \mathcal{X} \rightarrow M$ is defined analogously: the action morphism in **(a1)** is replaced by

$$\mathcal{G} \times_{\mathbf{s}, M, \mathbf{a}} \mathcal{X} \longrightarrow \mathcal{X},$$

in such a way that $\mathbf{a}(gx) = \mathbf{t}(g)$ (cf. **(a2)**), and there are 2-isomorphisms (cf. **(a3)**)

$$(hg) \cdot x \xrightarrow{\beta} h \cdot (gx), \quad 1x \xrightarrow{\varepsilon} x$$

satisfying the following higher coherence conditions (cf. **(a4)**):

$$\begin{array}{ccc} (lhgx) : & (lh \cdot g)x \xrightarrow{\beta} lh \cdot gx \xrightarrow{\beta} l(h \cdot gx) & \\ & \alpha \cdot \text{id} \uparrow & \uparrow \text{id} \cdot \beta \\ & (l \cdot hg)x \xrightarrow{\beta} l(hg \cdot x) & \end{array}$$

$$\begin{array}{ccc} (g1x) : & g1 \cdot x \xrightarrow{\beta} g \cdot 1x & \\ \rho \cdot \text{id} \downarrow & \swarrow \text{id} \cdot \varepsilon & \\ & gx & \end{array} \quad \begin{array}{ccc} (1gx) : & 1g \cdot x \xrightarrow{\beta} 1 \cdot gx & \\ \lambda \cdot \text{id} \downarrow & \swarrow \varepsilon & \\ & gx & \end{array}$$

Similarly to (3.8), we have an action-projection map

$$\Delta : \mathcal{G} \times_{\mathbf{s}, M, \mathbf{a}} \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}, \quad (g, x) \mapsto (gx, x).$$

When M is a point, similar notions of action (with varying levels of strictness) have been considered e.g. in [4, 8, 20, 43, 54]. Categorical actions similar to Def. 3.15 are also studied in [28, Sec. 5] using simplicial methods.

We present here some examples, others will be discussed in Section 3.4.

Example 3.17 A cfg-groupoid $\mathcal{G} \rightrightarrows M$ acts on itself on the right and on the left by multiplication, with moment maps $\mathbf{s} : \mathcal{G} \rightarrow M$ and $\mathbf{t} : \mathcal{G} \rightarrow M$ respectively. The associativity and identity 2-isomorphisms of these actions are induced by the corresponding 2-isomorphisms of the groupoid.

Example 3.18 Given an action of \mathcal{G} on \mathcal{X} along $\mathbf{a} : \mathcal{X} \rightarrow M$ and a smooth map $\iota : N \rightarrow M$, let $\mathcal{X}_N = N \times_M \mathcal{X}$. Then there is natural action of \mathcal{G}_N (see Remark 3.4) on \mathcal{X}_N .

As a particular case, given a cfg-groupoid $\mathcal{G} \rightrightarrows M$, let \mathcal{G}_x be the cfg-group defined by its isotropy at $x \in M$, as in (3.2). Then the groupoid multiplication restricts to an action (on the right) of \mathcal{G}_x on $\mathbf{s}^{-1}(x)$.

Example 3.19 Let the cfg-groupoid $\mathcal{G} \rightrightarrows M$ act on the right on the categories fibred in groupoids \mathcal{X}_1 e \mathcal{X}_2 , along the morphisms $\mathbf{a}_1 : \mathcal{X}_1 \rightarrow M$ and $\mathbf{a}_2 : \mathcal{X}_2 \rightarrow M$. The 2-isomorphisms associated with these actions are denoted by β_1, ε_1 , and β_2, ε_2 , respectively. Then there is an induced action over $\mathcal{X}_1 \times_M \mathcal{X}_2$ given by $(x_1, x_2)g = (x_1g, x_2g)$. The associativity 2-isomorphism is defined by

$$(x_1, x_2) \cdot gh = (x_1 \cdot gh, x_2 \cdot gh) \xrightarrow{\beta_1 \times \beta_2} (x_1g \cdot h, x_2g \cdot h) = (x_1, x_2)g \cdot h,$$

and the identity 2-isomorphism is defined by

$$(x_1, x_2)1 = (x_11, x_21) \xrightarrow{\varepsilon_1 \times \varepsilon_2} (x_1, x_2).$$

3.3 Equivariant morphisms

For objects in $\text{CFG}_{\mathcal{C}}$ endowed with actions, we consider a natural notion of equivariant morphism.

Definition 3.20 Let $\mathcal{G} \rightrightarrows M$ be a cfg-groupoid acting (on the right) on $\mathcal{X}_i \in \text{Obj}(\text{CFG}_{\mathcal{C}})$ (with action map \mathbf{a}_i), along the map $\mathbf{a}_i : \mathcal{X}_i \rightarrow M$, $i = 1, 2$. A morphism $F : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ is **\mathcal{G} -equivariant** if $\mathbf{a}_2 F = \mathbf{a}_1$ and there is a given 2-isomorphism $\delta : \mathbf{a}_2 \circ (F \times \text{id}) \xrightarrow{\sim} F \circ \mathbf{a}_1$ that makes the square

$$\begin{array}{ccc} \mathcal{X}_1 \times_M \mathcal{G} & \xrightarrow{\mathbf{a}_1} & \mathcal{X}_1 \\ F \times \text{id} \downarrow & & \downarrow F \\ \mathcal{X}_2 \times_M \mathcal{G} & \xrightarrow{\mathbf{a}_2} & \mathcal{X}_2 \end{array}$$

2-commute, and that satisfies the higher coherence conditions expressed by the commutativity of the following diagrams (with the corresponding labels displayed on the left):

$$(\delta\beta_1\beta_2) : \begin{array}{ccccc} F(x) \cdot g_1 g_2 & \xrightarrow{\beta_2} & F(x)g_1 \cdot g_2 & \xrightarrow{\delta \cdot \text{id}} & F(xg_1)g_2 \\ \delta \downarrow & & & & \downarrow \delta \\ F(x \cdot g_1 g_2) & \xrightarrow{F(\beta_1)} & F(xg_1 \cdot g_2) & & \end{array}$$

$$(\delta\varepsilon_1\varepsilon_2) : \begin{array}{ccc} F(x) \cdot 1 & \xrightarrow{\varepsilon_2} & F(x) \\ \delta \downarrow & \nearrow F(\varepsilon_1) & \\ F(x \cdot 1) & & \end{array}$$

where $x \in \mathcal{X}_1$, $g_1, g_2 \in \mathcal{G}$ are such that the compositions make sense, 1 is the appropriate groupoid identity, β_1, β_2 are the associativity 2-isomorphisms of the actions, and $\varepsilon_1, \varepsilon_2$ are the identity 2-isomorphisms of the actions (see Def. 3.15). We will often refer to δ as the equivariance 2-isomorphism.

Clearly, the identity morphism is always \mathcal{G} -equivariant with $\delta = \text{id}_{\mathbf{a}}$. The following property is also naturally expected.

Lemma 3.21 The composition of \mathcal{G} -equivariant morphisms is \mathcal{G} -equivariant.

PROOF: Given two equivariant morphisms $\mathcal{X}_1 \xrightarrow{F_1} \mathcal{X}_2 \xrightarrow{F_2} \mathcal{X}_3$, with corresponding 2-isomorphisms δ_1 and δ_2 (as in Def. 3.20), the composition $F_2 F_1$ is equivariant with respect to

$$\delta = (\text{id}_{F_2} \circ \delta_1) * (\delta_2 \circ \text{id}_{F_1 \times \text{id}}).$$

More explicitly, for any $(x, g) \in \mathcal{X} \times_M \mathcal{G}$, we have

$$\delta : F_2 F_1(x) \cdot g \xrightarrow{\delta_2} F_2(F_1(x) \cdot g) \xrightarrow{F_2(\delta_1)} F_2 F_1(x \cdot g).$$

Higher coherences $(\delta\beta_1\beta_2)$ and $(\delta\varepsilon_1\varepsilon_2)$ for δ follow from the respective coherences for δ_1 and δ_2 . \square

3.4 Stacky principal bundles

Let $\mathcal{G} \rightrightarrows M$ be a cfg-groupoid acting (on the right) on $\mathcal{X} \in \text{Obj}(\text{CFG}_C)$ along $\mathbf{a} : \mathcal{X} \rightarrow M$. Let $\mathcal{S} \in \text{Obj}(\text{CFG}_C)$ and $\mathbf{r} : \mathcal{X} \rightarrow \mathcal{S}$ be a morphism. We consider a natural compatibility between the action of \mathcal{G} on \mathcal{X} and the morphism \mathbf{r} (see Section 2.4 for a similar notion in the context of Lie groupoids).

Definition 3.22 *We say that \mathcal{G} acts on the fibers of $\mathbf{r} : \mathcal{X} \rightarrow \mathcal{S}$ if there is a given 2-isomorphism $\gamma : \mathbf{r} \circ \text{pr} \rightarrow \mathbf{r} \circ \mathbf{A}$ that makes the square*

$$\begin{array}{ccc} \mathcal{X} \times_M \mathcal{G} & \xrightarrow{\mathbf{A}} & \mathcal{X} \\ \text{pr} \downarrow & & \downarrow \mathbf{r} \\ \mathcal{X} & \xrightarrow{\mathbf{r}} & \mathcal{S} \end{array} \quad (3.9)$$

2-commute and that satisfies the higher coherence conditions expressed by the commutativity of the following diagrams (with the corresponding labels displayed on the left):

$$\begin{array}{ccc} (\gamma\beta) : & \begin{array}{ccc} \mathbf{r}(x) & \xrightarrow{\gamma} & \mathbf{r}(xg) \\ \gamma \downarrow & & \downarrow \gamma \\ \mathbf{r}(x(gh)) & \xrightarrow{\mathbf{r}(\beta)} & \mathbf{r}((xg)h), \end{array} & (\gamma\varepsilon) : \begin{array}{ccc} \mathbf{r}(x) & & \\ \gamma \downarrow & \searrow \text{id} & \\ \mathbf{r}(x \cdot 1) & \xrightarrow{\mathbf{r}(\varepsilon)} & \mathbf{r}(x), \end{array} \end{array}$$

where $x \in \mathcal{X}$ and $g, h \in \mathcal{G}$ are such that the compositions make sense, $1 \in \mathcal{G}$ denotes the appropriate identity, β is the associativity 2-isomorphism of the action, and ε is the identity 2-isomorphism of the action (see (a3)). We will say that \mathcal{G} acts on the fibers of \mathbf{r} via γ .

A similar definition holds for left actions.

Note that, for any cfg-groupoid $\mathcal{G} \rightrightarrows M$, the action by right multiplication is on the fibers of $\mathbf{t} : \mathcal{G} \rightarrow M$, and the action by left multiplication is on the fibers of $\mathbf{s} : \mathcal{G} \rightarrow M$.

If \mathcal{G} acts on the fibers of $\mathbf{r} : \mathcal{X} \rightarrow \mathcal{S}$, we may consider the morphism

$$\mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X} \times_{\mathcal{S}} \mathcal{X}, \quad (x, g) \mapsto (x, \gamma(x, g), xg). \quad (3.10)$$

Remark 3.23

- (a) If $\mathcal{S} = S$ is a manifold, then the condition for an action to be on the fibers of $r : \mathcal{X} \rightarrow S$ is just the commutativity of the square (3.9), as in this case the 2-isomorphism γ is necessarily equal to the identity and the higher coherences $(\gamma\beta)$, $(\gamma\varepsilon)$ are automatically satisfied (cf. Remark 3.2).
- (b) If \mathcal{G} acts on the fibers of r via γ and $r' : \mathcal{S} \rightarrow \mathcal{S}'$ is a morphism, then \mathcal{G} acts on the fibers of $r'r$ via $\text{id}_{r'} \circ \gamma$.

We will be mostly interested in the context of differentiable stacks. Let $\mathcal{G} \rightrightarrows M$ be a stacky Lie groupoid acting (on the right) on a differentiable stack \mathcal{X} along the map $a : \mathcal{X} \rightarrow M$.

Definition 3.24 *We say that a morphism $r : \mathcal{X} \rightarrow \mathcal{S}$ defines a (right) **principal \mathcal{G} -bundle** \mathcal{X} over a differentiable stack \mathcal{S} if the following conditions are satisfied:*

1. $r : \mathcal{X} \rightarrow \mathcal{S}$ is an epimorphism and a submersion,
2. \mathcal{G} acts on the fibers of r ,
3. The morphism $\mathcal{X} \times_M \mathcal{G} \longrightarrow \mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ (see (3.10)) is an isomorphism.

*Principal **left** \mathcal{G} -bundles are defined similarly.*

Note that if the differentiable stacks \mathcal{G} , \mathcal{X} and \mathcal{S} are representable, the previous definition recovers the notion of principal bundle in the smooth category.

Remark 3.25 Other viewpoints to smooth principal bundles naturally extend to higher settings. For example, higher principal bundles are often studied through methods of homotopy theory [29, 36, 19], generalizing the fact that smooth principal bundles can be defined (and classified) by maps into classifying spaces BG of Lie group(oid)s by means of (homotopical) pullbacks. For 2-groups, descriptions of principal bundles in terms of local trivializations and cocycles can be found e.g. in [3, 4, 54] ([38] reconciles these viewpoints with Def. 3.24 above).

Example 3.26 The target morphism $t : \mathcal{G} \rightarrow M$ of a stacky Lie groupoid is a principal right \mathcal{G} -bundle under the action of multiplication on the right. The same is true for $s : \mathcal{G} \rightarrow M$ and left multiplication.

Example 3.27 Consider a (right) action of a Lie groupoid $G \rightrightarrows G_0$ on a manifold X along $a : X \rightarrow G_0$. Then one may verify that X is a principal G -bundle over the quotient stack $[X/G]$.

In particular, any Lie groupoid $G \rightrightarrows G_0$ is such that G_0 is a principal G -bundle over BG .

Example 3.28 Consider Lie groupoids $X \rightrightarrows X_0$ and $Y \rightrightarrows Y_0$, and a right principal X - Y - bibundle P , i.e., a Hilsum-Skandalis bibundle presenting a morphism $BX \rightarrow BY$. Note that the surjective submersion $P \rightarrow X_0$ naturally gives rise to a morphism of quotient stacks

$$[X \backslash P] \rightarrow [X \backslash X_0] = BX$$

which is a submersion and an epimorphism. One can check that Y acts on $[X \backslash P]$ on the fibres of this map, and makes $[X \backslash P]$ into a principal Y -bundle over BX .

Example 3.29 When \mathcal{G} is a 2-group as in Example 3.6, principal \mathcal{G} -bundles (with strict actions as in [38, Sec. 6]) arise in the description of (non-abelian) *gerbes*, see [38, Sect. 7] (cf. [8, 9], see also [21]). A simple example is the correspondence between S^1 -gerbes and principal BS^1 -bundles over a manifold M (or, more generally, over a differentiable stack). Recall that one way to model S^1 -gerbes is via S^1 -central extensions (see e.g. [6]),

$$\begin{array}{ccccc} S^1 \times X_0 & \xrightarrow{\iota} & X & \longrightarrow & Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X_0 & \longrightarrow & X_0 & \longrightarrow & X_0. \end{array}$$

In such a case, the multiplication of $X \rightrightarrows X_0$ and the embedding ι give us a map

$$\begin{array}{ccc} X \times S^1 & \longrightarrow & X \\ \Downarrow & & \Downarrow \\ X_0 \times pt & \longrightarrow & X_0 \end{array}$$

which defines an action of BS^1 on $BX = [X_0/X]$. The fact that X is an S^1 -principal bundle over Y allows one to verify that this BS^1 -action makes BX into a principal BS^1 -bundle over $BY = [X_0/Y]$, as in Def. 3.24. This picture in fact extends to non-abelian gerbes [27]: in this case, a G -gerbe over a manifold M is a principal \mathcal{G} -bundle over M , where \mathcal{G} is the 2-group defined by the crossed module $G \rightarrow \text{Aut}(G)$.

To construct more examples of principal bundles, let us consider a cfg-groupoid $\mathcal{G} \rightrightarrows M$ acting on $\mathcal{X} \in \text{Obj}(\text{CFG}_{\mathcal{C}})$ on the fibers of $r : \mathcal{X} \rightarrow \mathcal{S}$, where \mathcal{S} is in $\text{Obj}(\text{CFG}_{\mathcal{C}})$. The next two propositions can be directly verified.

Proposition 3.30 *Consider a map of categories fibred in groupoids $\mathcal{S}' \rightarrow \mathcal{S}$ and the fibred product $\mathcal{X}' = \mathcal{X} \times_{\mathcal{S}} \mathcal{S}'$. The following holds:*

1. *There is an induced action of \mathcal{G} on \mathcal{X}' on the fibres of the natural projection $r' : \mathcal{X}' \rightarrow \mathcal{S}'$ (the “action on the first coordinate”).*
2. *If r is an epimorphism, so is r' .*
3. *If the map $\mathcal{G} \times_M \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ (3.10) is an isomorphism, then so is the corresponding map $\mathcal{G} \times_M \mathcal{X}' \rightarrow \mathcal{X}' \times_{\mathcal{S}'} \mathcal{X}'$.*
4. *If \mathcal{G} , \mathcal{X} , \mathcal{S} and \mathcal{S}' are differentiable stacks and r is a submersion, then so is r' .*

It follows that if $r : \mathcal{X} \rightarrow \mathcal{S}$ is a principal \mathcal{G} -bundle, then so is $r' : \mathcal{X}' \rightarrow \mathcal{S}'$.

Example 3.31 For a stacky Lie groupoid $\mathcal{G} \rightrightarrows M$, it follows from the previous proposition that left multiplication restricts to a principal \mathcal{G} -bundle $s^{-1}(x) \rightarrow \{x\}$, for each $x \in M$.

Example 3.32 Consider a morphism $F : \mathcal{X} \rightarrow \mathcal{Y}$ of differentiable stacks, and assume that \mathcal{Y} is presented by $Y \rightrightarrows Y_0$. Following Example 3.27, we know that Y_0 is principal Y -bundle over \mathcal{Y} , so it follows from the previous proposition that F induces a principal Y -bundle over \mathcal{X} .

Proposition 3.33 *Consider a smooth map $f : N \rightarrow M$, let \mathcal{G}_N and \mathcal{X}_N be as in Example 3.18. The following holds:*

1. *The action of \mathcal{G}_N on \mathcal{X}_N is on the fibres of the natural map $r_N : \mathcal{X}_N \rightarrow \mathcal{S}$.*
2. *If the map $\mathcal{G} \times_M \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ (3.10) is an isomorphism, then so is the corresponding map $\mathcal{G}_N \times_N \mathcal{X}_N \rightarrow \mathcal{X}_N \times_{\mathcal{S}} \mathcal{X}_N$.*
3. *If $\mathcal{X} \rightarrow \mathcal{S}$ is a principal \mathcal{G} -bundle, \mathcal{G}_N is a differentiable stack and r_N is a submersion and an epimorphism, then $r_N : \mathcal{X}_N \rightarrow \mathcal{S}$ is a principal \mathcal{G}_N -bundle.*

Example 3.34 For an étale stacky Lie groupoid $\mathcal{G} \rightrightarrows M$, fix $x \in M$ and consider the stacky Lie group \mathcal{G}_x . Suppose, for simplicity, that \mathcal{G} is transitive (see Cor. 3.12). It follows from the previous proposition that right multiplication on \mathcal{G} restricts to a principal \mathcal{G}_x -bundle $t : s^{-1}(x) \rightarrow M$.

Our last result in this section gives the expected relation between Examples 3.28 and 3.32 above.

Proposition 3.35 *Let \mathcal{X} and \mathcal{Y} be differentiable stacks, let $X \rightrightarrows X_0$ and $Y \rightrightarrows Y_0$ be Lie groupoids presenting them, and let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism. Let P be a right principal X - Y -bibundle presenting F . Then $[X \setminus P]$ fits into a canonical 2-cartesian diagram*

$$\begin{array}{ccc} [X \setminus P] & \longrightarrow & Y_0 \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{F} & \mathcal{Y}, \end{array}$$

in such a way that the resulting isomorphism $[X \setminus P] \cong \mathcal{X} \times_{\mathcal{Y}} Y_0$ is an identification of Y -principal bundles over \mathcal{X} .

PROOF: We start by replacing $X \rightrightarrows X_0$ with a Morita equivalent Lie groupoid, denoted by $X \ltimes P \rtimes Y \rightrightarrows P$, whose arrows are triples (x, z, y) in $X \times P \times Y$ such that $s(x) = a_0(z)$ and $t(y) = b_0(z)$; source and target maps are

$$s(x, z, y) = z, \quad t(x, z, y) = xzy,$$

and multiplication is defined by

$$(x, z, y)(\bar{x}, \bar{z}, \bar{y}) = (x\bar{x}, \bar{z}, \bar{y}y),$$

where $z = \bar{x}\bar{z}\bar{y}$ is the composability condition. A Morita $(X \ltimes P \rtimes Y)$ - X bibundle is given by $P \times_{X_0} X$, with left action $(x, z, y) \cdot (z, \bar{x}) = (xzy, x\bar{x})$ and right action $(z, \bar{x})x = (z, \bar{x}x)$. Moreover, the composition of this Morita bibundle with P gives rise to a right principal $(X \ltimes P \rtimes Y)$ - Y bibundle, which in fact corresponds to a morphism of groupoids,

$$\begin{array}{ccc} X \ltimes P \rtimes Y & \xrightarrow{b} & Y \\ \Downarrow & & \Downarrow \\ P & \xrightarrow{b_0} & Y_0, \end{array}$$

explicitly given by $b(x, z, y) = y^{-1}$. In other words, this map b is a presentation for $F : \mathcal{X} \rightarrow \mathcal{Y}$ upon the identification $B(X \ltimes P \rtimes Y) \cong BX = \mathcal{X}$.

We will now compute the (weak) fibred product of groupoids

$$\begin{array}{ccc} W & \longrightarrow & Y_0 \\ \downarrow & & \downarrow \\ X \ltimes P \rtimes Y & \xrightarrow{b} & Y, \end{array} \quad (3.11)$$

where Y_0 denotes the trivial groupoid $Y_0 \rightrightarrows Y_0$, and prove that there is an identification $BW = B(X \ltimes P)$. Since $B(X \ltimes P) = [X \backslash P]$, this leads to an isomorphism $[X \backslash P] \cong \mathcal{X} \times_{\mathcal{Y}} Y_0$ (by Prop. 2.14).

The fibred product W in (3.11) is described by objects $(z, y, y_0) \in P \times Y \times Y_0$ such that $s(y) = b_0(z)$ and $t(y) = y_0$, and arrows from (z, y, y_0) to $(\bar{z}, \bar{y}, \bar{y}_0)$ given by pairs $(x, y_1) \in X \times Y$ such that

$$s(x) = a_0(z), \quad t(y_1) = b_0(z), \quad xzy_1 = \bar{z}, \quad \bar{y} = yy_1,$$

assuming that $\bar{y}_0 = y_0$ (otherwise there are no arrows). The multiplication between $(\bar{x}, \bar{y}_1) : (z, y, y_0) \rightarrow (\bar{z}, \bar{y}, y_0)$ and $(x, y_1) : (\bar{z}, \bar{y}, y_0) \rightarrow (\tilde{z}, \tilde{y}, y_0)$ is defined componentwise:

$$(x, y_1)(\bar{x}, \bar{y}_1) = (x\bar{x}, \bar{y}_1y_1).$$

We note that W admits an alternative description: objects are pairs (z, y) such that $s(y) = b_0(z)$, arrows are given by (x, z, y, y_1) such that $s(x) = a_0(z)$, $t(y_1) = b_0(z) = s(y)$, and source and target maps are:

$$s(x, z, y, y_1) = (z, y), \quad t(x, z, y, y_1) = (xzy_1, yy_1).$$

For multiplication, we have

$$(x, z, y, y_1)(\bar{x}, \bar{z}, \bar{y}, \bar{y}_1) = (x\bar{x}, \bar{z}, \bar{y}, \bar{y}_1y_1),$$

where $(\bar{x}\bar{z}\bar{y}_1, \bar{y}\bar{y}_1) = (z, y)$ is the condition for composability.

Using this alternative description of W , we define a groupoid morphism

$$c : W \longrightarrow X \ltimes P \quad (3.12)$$

as follows: on objects, we let

$$c_0(z, y) = zy^{-1},$$

and on arrows we set

$$c(x, z, y, y_1) = (x, zy^{-1}).$$

We leave to the reader to check that c_0 is a surjective submersion (which follows from P being right principal), and that c , as groupoid morphism, is fully faithful. This shows that c is a weak equivalence, hence induces the desired identification.

To see that the identification

$$BW \cong \mathcal{X} \times_{\mathcal{Y}} Y_0 \xrightarrow{\sim} [X \backslash P] \quad (3.13)$$

induced by c is Y -equivariant, we first notice that the Y -action on $BW \cong \mathcal{X} \times_Y Y_0$ is presented by the groupoid morphism $W \times_{Y_0} Y \rightarrow W$ given on objects and arrows by

$$((z, y), \bar{y}) \mapsto (z, \bar{y}^{-1}y), \quad ((x, z, y, y_1), \bar{y}) \mapsto (x, z, \bar{y}^{-1}y, y_1),$$

respectively. Similarly, the Y -action on $[X \setminus P]$ is presented by the groupoid morphism $(X \ltimes P) \times_{Y_0} Y \rightarrow (X \ltimes P)$ where Y acts on the P factor only. (Here $W \times_{Y_0} Y$ and $(X \ltimes P) \times_{Y_0} Y$ denote the fibred product of groupoids viewing Y as the trivial groupoid $Y \rightrightarrows Y$). One may directly check that c (3.12) is Y -equivariant, which implies the same property for (3.13). \square

4 Quotients

Let a cfg-groupoid $\mathcal{G} \rightrightarrows M$ act on $\mathcal{X} \in \text{Obj}(\text{CFG}_{\mathcal{C}})$. Our goal in this section is to make appropriate sense of the quotient of \mathcal{X} by the \mathcal{G} -action. We start by defining a natural category fibred in groupoids $\mathcal{X}/_p \mathcal{G}$ associated with the action, referred to as the *prequotient*; its stackification will then be defined as the *quotient* stack \mathcal{X}/\mathcal{G} .

4.1 The (pre)quotient of an action

The definition of the prequotient given below is a natural generalization of the construction given in [8] when M is a point.

For a (right) action of a cfg-groupoid $\mathcal{G} \rightrightarrows M$ on a category fibred in groupoids \mathcal{X} along $\mathbf{a} : \mathcal{X} \rightarrow M$, we will define a category $\mathcal{X}/_p \mathcal{G}$ and a functor $\pi_{\mathcal{X}/_p \mathcal{G}} : \mathcal{X}/_p \mathcal{G} \rightarrow \mathcal{C}$ to the category of manifolds as follows. The objects of $\mathcal{X}/_p \mathcal{G}$ are given by

$$\text{Obj}(\mathcal{X}/_p \mathcal{G}) := \text{Obj}(\mathcal{X}). \quad (4.1)$$

For $x, y \in \text{Obj}(\mathcal{X})$, the set of morphisms in $\mathcal{X}/_p \mathcal{G}$ between x and y is

$$\text{Hom}_{\mathcal{X}/_p \mathcal{G}}(x, y) := \{(g, b) : g \in \mathcal{G}, \mathbf{a}(x) = \mathbf{t}(g), b : x \cdot g \rightarrow y \text{ in } \mathcal{X}\} / \sim \quad (4.2)$$

where the equivalence relation \sim results from identifying two pairs (g, b) and (\bar{g}, \bar{b}) if there exists $j : g \xrightarrow{\sim} \bar{g}$ in \mathcal{G} such that $\mathbf{t}(j) = \text{id}_{\mathbf{a}(x)}$ and the following triangle commutes:

$$\begin{array}{ccc} x \cdot g & & \\ \text{id}_x \cdot j \downarrow & \searrow b & \\ x \cdot \bar{g} & \xrightarrow{\bar{b}} & y. \end{array} \quad (4.3)$$

Considering the structure functors $\pi_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{C}$ and $\pi_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{C}$, note that $\pi_{\mathcal{G}}(g) = \pi_{\mathcal{G}}(\bar{g}) = \pi_{\mathcal{X}}(x)$ and $\pi_{\mathcal{G}}(j) = \text{id}_{\pi_{\mathcal{X}}(x)}$. We denote the equivalence class of (g, b) in $\text{Hom}_{\mathcal{X}/_p \mathcal{G}}(x, y)$ by $[g, b]$. The composition of morphisms,

$$x \xrightarrow{[g, b]} y \xrightarrow{[h, c]} z,$$

is defined as follows. Let $\mu : U \rightarrow V$ be the morphism in \mathcal{C} given by $\pi_{\mathcal{X}}(b)$. Since $\pi_{\mathcal{G}}(h) = V$, we may take a morphism $\mu_h : \mu^* h \rightarrow h$ in \mathcal{G} over μ representing the pull back of h along μ (this involves a choice, unique up to canonical isomorphism). Since $\mathbf{a}(y) = \mathbf{t}(h)$ and M is fibred in

sets, it follows that $\mathbf{t}(\mu^*h) = \mathbf{s}(g)$ and $\mathbf{a}(b) = \mathbf{t}(\mu_h)$. Hence it makes sense to compose $[h, c]$ and $[g, b]$ by

$$[h, c][g, b] := [g \cdot \mu^*h, c \circ (b \cdot \mu_h) \circ \beta(x, g, \mu^*h)], \quad (4.4)$$

where β is defined in **(a3)** of Section 3.2, and this definition does not depend on the choices of representatives (g, b) and (h, c) , nor on the choices of μ^*h and μ_h . For each $x \in \text{Obj}(\mathcal{X})$, there is an associated identity $\text{id}_x \in \text{Hom}_{\mathcal{X}/_p\mathcal{G}}(x, x)$ given by

$$\text{id}_x := [1_{\mathbf{a}(x)}, \varepsilon(x)], \quad (4.5)$$

with ε defined in **(a3)**. Proposition 4.1 below asserts that $\mathcal{X}/_p\mathcal{G}$ is indeed a category, with objects (4.1) and morphisms (4.2). Moreover, the maps $\text{Obj}(\mathcal{X}/_p\mathcal{G}) \rightarrow \text{Obj}(\mathcal{C})$, sending objects x to $\pi_{\mathcal{X}}(x)$, and $\text{Hom}_{\mathcal{X}/_p\mathcal{G}}(x, y) \rightarrow \text{Hom}_{\mathcal{C}}(\pi_{\mathcal{X}}(x), \pi_{\mathcal{X}}(y))$, given by

$$[g, b] \mapsto \pi_{\mathcal{X}}(b) \quad (4.6)$$

(the reader may verify that this is well defined), define a functor

$$\pi_{\mathcal{X}/_p\mathcal{G}} : \mathcal{X}/_p\mathcal{G} \longrightarrow \mathcal{C}. \quad (4.7)$$

Proposition 4.1 *The definitions in (4.1), (4.2), (4.4) and (4.5) make $\mathcal{X}/_p\mathcal{G}$ into a category and $\pi_{\mathcal{X}/_p\mathcal{G}}$ into a functor, in such a way that $(\mathcal{X}/_p\mathcal{G}, \pi_{\mathcal{X}/_p\mathcal{G}})$ is a category fibred in groupoids over \mathcal{C} .*

The proof of this proposition is discussed in Appendix A.3.

Definition 4.2 *The category fibred in groupoids $\mathcal{X}/_p\mathcal{G}$ is the **prequotient** of \mathcal{X} by the action of \mathcal{G} . The **quotient** \mathcal{X}/\mathcal{G} of \mathcal{X} by the action of \mathcal{G} is the stackification of the prequotient $\mathcal{X}/_p\mathcal{G}$.*

There is a natural projection functor

$$\mathbf{q} : \mathcal{X} \longrightarrow \mathcal{X}/_p\mathcal{G} \quad (4.8)$$

defined by

$$\mathbf{q}(x) := x, \quad \mathbf{q}(b) := [1_{\mathbf{a}(x)}, b \circ \varepsilon(x)], \quad (4.9)$$

for $x \in \text{Obj}(\mathcal{X})$ and $b : x \rightarrow y$ morphism in \mathcal{X} . The fact that \mathbf{q} respects composition follows from $(xg1)$ in **(a4)**, while \mathbf{q} respects the identity by definition. Since $\pi_{\mathcal{X}}(\varepsilon) = \text{id}$, it follows that \mathbf{q} commutes with the projections to \mathcal{C} . This shows the next result.

Proposition 4.3 *The functor \mathbf{q} is a morphism of categories fibred in groupoids.*

We keep the notation

$$\mathbf{q} : \mathcal{X} \rightarrow \mathcal{X}/\mathcal{G} \quad (4.10)$$

for the morphism induced by (4.8) and the stackification $\mathcal{X}/_p\mathcal{G} \rightarrow \mathcal{X}/\mathcal{G}$ (see Prop. 2.2).

Remark 4.4 Note that any action of \mathcal{G} on \mathcal{X} is on the fibers of the prequotient map $\mathcal{X} \rightarrow \mathcal{X}/_p\mathcal{G}$ via γ_0 defined by

$$\gamma_0(x, g) := [g, \text{id}_{xg}] : x \rightarrow xg.$$

Here the higher coherences $(xg1)$ and $(x1g)$ of **(a4)** are used to verify that γ_0 is indeed a natural transformation, and $(xg1)$ is used to prove conditions $(\gamma\beta)$ and $(\gamma\varepsilon)$ of Def. 3.22. It follows from Remark 3.23(b) that \mathcal{G} also acts on the fibers of $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}$.

Example 4.5 (The representable case) Suppose that \mathcal{G} and \mathcal{X} are representable, i.e., isomorphic to manifolds G and X . Then the action of \mathcal{G} on \mathcal{X} boils down to an ordinary Lie groupoid action of $G \rightrightarrows M$ on X , along $a : X \rightarrow M$. Let us denote M by G_0 .

In this case, the prequotient $X/_p G$ is isomorphic in $\text{CFG}_{\mathcal{C}}$ to the category fibred in groupoids $[X/G]_p$, whose objects are trivial principal G -bundles equipped with an equivariant map to X , and whose morphisms are morphisms of bundles commuting with the map to X . Indeed, an object of $[X/G]_p$ is equivalent to a manifold U (thought of as the base of the bundle) equipped with a morphism $c : U \rightarrow X$. More explicitly, the corresponding G -bundle is

$$U \times_{ac, G_0, t} G \xrightarrow{\text{pt}} U,$$

and the map $U \times_{G_0} G \rightarrow X$ is given by $(z, g) \mapsto c(z) \cdot g$. Viewing X as a category fibred in groupoids, the pair (U, c) is the same as an object in X , and hence an object in $X/_p G$ (since $\text{Obj}(X/_p G) = \text{Obj}(X)$).

Similarly, a morphism in $[X/G]_p$ between the objects x, x' , corresponding to $c : U \rightarrow X$ and $c' : U' \rightarrow X$, is equivalent to a pair of smooth maps $\mu : U \rightarrow U'$ and $\nu : U \rightarrow G$ such that $ac = t\nu$ and $c \cdot \nu = c'\mu$ (here $c \cdot \nu$ is defined by the G -action on X). More explicitly, the bundle map $U \times_{G_0} G \rightarrow U' \times_{G_0} G$ is given by $(z, g) \mapsto (\mu(z), \nu(z)^{-1}g)$. We must show that giving such μ and ν is the same as giving a morphism in $X/_p G$ between x and x' . This follows from the definition of prequotient. Indeed, an object g of G (viewed in $\text{CFG}_{\mathcal{C}}$) is the same as a smooth map $\nu : \tilde{U} \rightarrow G$, and by (4.2) we have the condition $ac = t\nu$, which implies that $\tilde{U} = U$. The object $x \cdot g$ of X is given by $c \cdot \nu : U \rightarrow X$, and a morphism $b : x \cdot g \rightarrow x'$ in X is the same as a smooth map $\mu : U \rightarrow U'$ such that $c'\mu = c \cdot \nu$.

Following Def. 4.2, the quotient of X by G , given by the stackification of $[X/G]_p$, is just the usual quotient stack of an action of a Lie groupoid, as described in Section 2.4.

4.2 Conditions for the prequotient to be a prestack

Let a cfg-groupoid $\mathcal{G} \rightrightarrows M$ act on a category fibred in groupoids \mathcal{X} (on the right, along $\mathbf{a} : \mathcal{X} \rightarrow M$), and consider the prequotient $\mathcal{X}/_p \mathcal{G} \in \text{Obj}(\text{CFG}_{\mathcal{C}})$. We will be particularly interested in the case where $\mathcal{X}/_p \mathcal{G}$ is a prestack. The following are simple consequences of this fact:

Proposition 4.6 *Suppose that the prequotient $\mathcal{X}/_p \mathcal{G} \in \text{Obj}(\text{CFG}_{\mathcal{C}})$ is a prestack. Then*

- (a) *the morphism $\mathbf{q} : \mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}$ in (4.10) is an epimorphism;*
- (b) *the natural morphism $\mathcal{X} \times_{\mathcal{X}/_p \mathcal{G}} \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{X}/\mathcal{G}} \mathcal{X}$ is an isomorphism.*

PROOF: The morphism $\mathcal{X} \rightarrow \mathcal{X}/_p \mathcal{G}$ is surjective at the level of objects, hence it is an epimorphism. The morphism $\mathcal{X}/_p \mathcal{G} \rightarrow \mathcal{X}/\mathcal{G}$ is an epimorphism, see Prop. 2.2 (iii), and part (a) follows. Part (b) is standard. \square

We will now discuss conditions guaranteeing that the prequotient $\mathcal{X}/_p \mathcal{G}$ is a prestack.

Definition 4.7 *The action of \mathcal{G} on \mathcal{X} is said to be **1-free** if, for all $x \in \text{Obj}(\mathcal{X})$, the section functor of the action,*

$$\mathbf{A}(x, -) : \mathbf{t}^{-1}(\mathbf{a}(x)) \rightarrow \mathcal{X},$$

is faithful. (Here $\mathbf{t}^{-1}(\mathbf{a}(x))$ is the fiber of the functor \mathbf{t} over the object $\mathbf{a}(x)$.)

In other words, the action is 1-free if for all $j, j' : g \rightarrow \bar{g}$ in \mathcal{G} with $\mathbf{t}(j) = \mathbf{t}(j') = \text{id}_{\mathbf{a}(x)}$, we

$$\text{id}_x \cdot j = \text{id}_x \cdot j' \implies j = j'$$

(note the formal analogy with the usual set-theoretic notion of freeness, but now applied to 1-morphisms, hence the terminology). It is straightforward to verify that if the action-projection map Δ (3.8) of the action is faithful, then the action is 1-free.

The following is the main result of this section.

Proposition 4.8 *If \mathcal{G} is a stacky groupoid, \mathcal{X} is a prestack, and the action of \mathcal{G} on \mathcal{X} is 1-free (in particular, if the action-projection map is faithful), then $\mathcal{X}/_p \mathcal{G}$ is a prestack.*

PROOF: Consider a manifold U , an open cover $(U_\alpha \rightarrow U)_\alpha$, and consider objects $x, x' \in (\mathcal{X}/_p \mathcal{G})_U = \mathcal{X}_U$. We must verify the following:

(i) Given morphisms in $(\mathcal{X}/_p \mathcal{G})_U$,

$$[g, b], [\bar{g}, \bar{b}] : x \rightarrow x',$$

such that $[g, b]_{|U_\alpha} = [\bar{g}, \bar{b}]_{|U_\alpha}$ for each α , then $[g, b] = [\bar{g}, \bar{b}]$.

(ii) Given $[g_\alpha, b_\alpha] : x|_{U_\alpha} \rightarrow x'|_{U_\alpha}$ morphisms in $(\mathcal{X}/_p \mathcal{G})_{U_\alpha}$ such that $[g_\alpha, b_\alpha]_{|U_{\alpha\beta}} = [g_\beta, b_\beta]_{|U_{\alpha\beta}}$ for all α and β , then there exists $[g, b] : x \rightarrow x'$ in $(\mathcal{X}/_p \mathcal{G})_U$ such that, for all α , $[g, b]_{|U_\alpha} = [g_\alpha, b_\alpha]$.

Verifying (i) amounts to proving that there exists $j : g \xrightarrow{\sim} \bar{g}$ in \mathcal{G} such that $\mathbf{t}(j) = \text{id}_{\mathbf{a}(x)}$ and $\bar{b} \circ (\text{id}_x \cdot j) = b$. Here $g, \bar{g} \in \mathcal{G}$ and $\mathbf{a}(x) = \mathbf{t}(g) = \mathbf{t}(\bar{g})$; moreover, $b : x \cdot g \rightarrow x'$ and $\bar{b} : x \cdot \bar{g} \rightarrow x'$ are arrows in \mathcal{X} over the identity of U , hence isomorphisms. Notice also that $[g, b]_{|U_\alpha} = [g|_{U_\alpha}, b|_{U_\alpha}]$ by Remark A.8.

The assumption that $[g, b]_{|U_\alpha} = [\bar{g}, \bar{b}]_{|U_\alpha}$ says that, for all α , there is a $j_\alpha : g|_{U_\alpha} \xrightarrow{\sim} \bar{g}|_{U_\alpha}$ such that $\mathbf{t}(j_\alpha) = \text{id}_{\mathbf{a}(x|_{U_\alpha})}$ and

$$\bar{b}|_{U_\alpha} \circ (\text{id}_{x|_{U_\alpha}} \cdot j_\alpha) = b|_{U_\alpha}. \quad (4.11)$$

Restricting this last equality to $U_{\alpha\beta}$ and using the fact that \bar{b} is an isomorphism, it follows that, for all α and β ,

$$\text{id}_{x|_{U_{\alpha\beta}}} \cdot j_{\alpha|_{U_{\alpha\beta}}} = \text{id}_{x|_{U_{\alpha\beta}}} \cdot j_{\beta|_{U_{\alpha\beta}}}.$$

Since the action is 1-free, it follows that $j_{\alpha|_{U_{\alpha\beta}}} = j_{\beta|_{U_{\alpha\beta}}}$, and since \mathcal{G} is a prestack it follows that there exists $j : g \rightarrow \bar{g}$ in \mathcal{G} such that $\pi_{\mathcal{G}}(j) = \text{id}_U$ and $j|_{U_\alpha} = j_\alpha$. Note that j is an isomorphism. Using that $\pi_{\mathcal{G}}(j) = \text{id}_U$ and M is fibred in sets, we see that $\mathbf{t}(j) = \text{id}_{\mathbf{a}(x)}$. By (4.11) and $j|_{U_\alpha} = j_\alpha$, we have that

$$(\bar{b} \circ (\text{id}_x \cdot j))_{|U_\alpha} = b|_{U_\alpha}.$$

Since \mathcal{X} is a prestack, it follows that $\bar{b} \circ (\text{id}_x \cdot j) = b$, and we conclude the proof of (i).

To prove (ii), we notice that the b_α 's are isomorphisms and that $[g_\alpha, b_\alpha]_{|U_{\alpha\beta}} = [g_\alpha|_{U_{\alpha\beta}}, b_\alpha|_{U_{\alpha\beta}}]$. Since $[g_\alpha, b_\alpha]_{|U_{\alpha\beta}} = [g_\beta, b_\beta]_{|U_{\alpha\beta}}$, there are $j_{\alpha\beta} : g_\beta|_{U_{\alpha\beta}} \xrightarrow{\sim} g_\alpha|_{U_{\alpha\beta}}$ in \mathcal{G} such that $\mathbf{t}(j_{\alpha\beta}) = \text{id}_{\mathbf{a}(x|_{U_{\alpha\beta}})}$ and

$$b_\alpha|_{U_{\alpha\beta}} \circ (\text{id}_{x|_{U_{\alpha\beta}}} \cdot j_{\alpha\beta}) = b_\beta|_{U_{\alpha\beta}}.$$

Restricting the above equality to $U_{\alpha\beta\gamma}$ and using the fact that the b_α 's are isomorphisms, we conclude that

$$\begin{aligned} \text{id}_{x|U_{\alpha\beta\gamma}} \cdot j_{\alpha\gamma|U_{\alpha\beta\gamma}} &= b_{\alpha|U_{\alpha\beta\gamma}}^{-1} \circ b_{\gamma|U_{\alpha\beta\gamma}} \\ &= b_{\alpha|U_{\alpha\beta\gamma}}^{-1} \circ b_{\beta|U_{\alpha\beta\gamma}} \circ b_{\beta|U_{\alpha\beta\gamma}}^{-1} \circ b_{\gamma|U_{\alpha\beta\gamma}} \\ &= \text{id}_{x|U_{\alpha\beta\gamma}} \cdot (j_{\alpha\beta|U_{\alpha\beta\gamma}} \circ j_{\beta\gamma|U_{\alpha\beta\gamma}}). \end{aligned}$$

It follows from the action being 1-free that $j_{\alpha\gamma|U_{\alpha\beta\gamma}} = j_{\alpha\beta|U_{\alpha\beta\gamma}} \circ j_{\beta\gamma|U_{\alpha\beta\gamma}}$, and using that \mathcal{G} is a stack it follows that there exists $g \in \mathcal{G}_U$ and isomorphisms

$$\varphi_\alpha : g|_{U_\alpha} \xrightarrow{\sim} g_\alpha$$

in \mathcal{G}_{U_α} such that

$$j_{\alpha\beta} \circ \varphi_\beta|_{U_{\alpha\beta}} = \varphi_\alpha|_{U_{\alpha\beta}}.$$

Notice that $\mathbf{t}(\varphi_\alpha) = \text{id}_{\mathbf{a}(x|_{U_\alpha})}$. We now use the fact that M , viewed as a category fibred in groupoids, is automatically a prestack to prove that

$$\mathbf{t}(g) = \mathbf{a}(x).$$

Indeed, for all α we have $(\mathbf{t}g)|_{U_\alpha} = \mathbf{t}(g|_{U_\alpha}) = \mathbf{t}(g_\alpha) = \mathbf{a}(x|_{U_\alpha}) = (\mathbf{a}x)|_{U_\alpha}$, where in the second equality we used the fact that M is fibred in sets.

Since $\mathbf{t}(\varphi_\alpha) = \text{id}_{\mathbf{a}(x|_{U_\alpha})}$, we can define an isomorphism in \mathcal{X}_{U_α} by

$$\bar{b}_\alpha := b_\alpha \circ (\text{id}_{x|_{U_\alpha}} \cdot \varphi_\alpha) : (x \cdot g)|_{U_\alpha} = x|_{U_\alpha} \cdot g|_{U_\alpha} \xrightarrow{\sim} x'|_{U_\alpha}.$$

We have that

$$\begin{aligned} \bar{b}_\alpha|_{U_{\alpha\beta}} &= b_\alpha|_{U_{\alpha\beta}} \circ (\text{id}_{x|_{U_{\alpha\beta}}} \cdot \varphi_\alpha|_{U_{\alpha\beta}}) \\ &= b_\alpha|_{U_{\alpha\beta}} \circ (\text{id}_{x|_{U_{\alpha\beta}}} \cdot (j_{\alpha\beta} \circ \varphi_\beta|_{U_{\alpha\beta}})) \\ &= b_\alpha|_{U_{\alpha\beta}} \circ (\text{id}_{x|_{U_{\alpha\beta}}} \cdot j_{\alpha\beta}) \circ (\text{id}_{x|_{U_{\alpha\beta}}} \cdot \varphi_\beta|_{U_{\alpha\beta}}) \\ &= b_\beta|_{U_{\alpha\beta}} \circ (\text{id}_{x|_{U_{\alpha\beta}}} \cdot \varphi_\beta|_{U_{\alpha\beta}}) \\ &= \bar{b}_\beta|_{U_{\alpha\beta}}. \end{aligned}$$

Since \mathcal{X} is a prestack, there exists a morphism in \mathcal{X}_U ,

$$b : x \cdot g \longrightarrow x',$$

such that $b|_{U_\alpha} = \bar{b}_\alpha$, for all α . This produces a morphism in $(\mathcal{X}/\mathcal{G})_U$:

$$[g, b] : x \longrightarrow x'.$$

The existence of the isomorphisms φ_α implies that $[g, b]|_{U_\alpha} = [g_\alpha, b_\alpha]$, and this concludes the proof of (ii). \square

4.3 The universal property of (pre)quotients

Let $\mathcal{G} \rightrightarrows M$ be a cfg-groupoid acting (on the right) on $\mathcal{X} \in \text{Obj}(\text{CFG}_{\mathcal{C}})$.

Proposition 4.9 *The quotient map $\mathbf{q} : \mathcal{X} \rightarrow \mathcal{X}/_p \mathcal{G}$ has the following universal property:*

- (i) *Suppose that \mathcal{G} acts on \mathcal{X} on the fibers of a morphism $\mathbf{r} : \mathcal{X} \rightarrow \mathcal{S}$ in $\text{CFG}_{\mathcal{C}}$ via γ (see Def. 3.22). Then there exists a pair (Φ, φ) , where*

$$\Phi : \mathcal{X}/_p \mathcal{G} \rightarrow \mathcal{S},$$

$\varphi : \Phi \mathbf{q} \xrightarrow{\sim} \mathbf{r}$, and such that

$$\gamma * (\varphi \circ \text{id}_{\text{pr}}) = (\varphi \circ \text{id}_{\mathbf{A}}) * (\text{id}_{\Phi} \circ \gamma_0),$$

where $\text{pr} : \mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X}$ is the natural projection, $\mathbf{A} : \mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X}$ is the action map, and γ_0 defined in Rem. 4.4. In other words, this last condition means that for any $(x, g) \in \mathcal{X} \times_M \mathcal{G}$ the following diagram of morphisms in \mathcal{S} commutes:

$$(\varphi\gamma) : \quad \begin{array}{ccc} \Phi(x) & \xrightarrow{\varphi} & \mathbf{r}(x) \\ \Phi[g, \text{id}_{xg}] \downarrow & & \downarrow \gamma \\ \Phi(xg) & \xrightarrow{\varphi} & \mathbf{r}(xg) \end{array}$$

(We notice that the left vertical map is $\Phi(\gamma_0)$.)

- (ii) *The pair (Φ, φ) is unique up to canonical isomorphism, in the following sense. Let \mathcal{G} act on the fibers of another morphism $\bar{\mathbf{r}} : \mathcal{X} \rightarrow \mathcal{S}$ via $\bar{\gamma}$, and let $\varrho : \mathbf{r} \rightarrow \bar{\mathbf{r}}$ be a 2-isomorphism such that for any $(x, g) \in \mathcal{X} \times_M \mathcal{G}$ the following diagram of morphisms in \mathcal{S} commutes:*

$$(\varrho\gamma) : \quad \begin{array}{ccc} \mathbf{r}(x) & \xrightarrow{\varrho} & \bar{\mathbf{r}}(x) \\ \gamma \downarrow & & \downarrow \bar{\gamma} \\ \mathbf{r}(xg) & \xrightarrow{\varrho} & \bar{\mathbf{r}}(xg). \end{array}$$

Let $(\bar{\Phi}, \bar{\varphi})$ be a pair where $\bar{\Phi} : \mathcal{X}/_p \mathcal{G} \rightarrow \mathcal{S}$, $\bar{\varphi} : \bar{\Phi} \mathbf{q} \xrightarrow{\sim} \bar{\mathbf{r}}$, and such that condition $(\varphi\gamma)$ is satisfied. Then there exists a unique $\psi : \Phi \xrightarrow{\sim} \bar{\Phi}$ such that, for any $x \in \mathcal{X}$, the following square commutes:

$$(\psi\varphi\varrho) : \quad \begin{array}{ccc} \Phi(x) & \xrightarrow{\psi} & \bar{\Phi}(x) \\ \varphi \downarrow & & \downarrow \bar{\varphi} \\ \mathbf{r}(x) & \xrightarrow{\varrho} & \bar{\mathbf{r}}(x). \end{array}$$

PROOF: We start with the existence of (Φ, φ) .

We define the morphism $\Phi : \mathcal{X}/_p \mathcal{G} \rightarrow \mathcal{S}$ as follows. For any $x \in \text{Obj}(\mathcal{X}/_p \mathcal{G}) = \text{Obj}(\mathcal{X})$, we set $\Phi(x) := \mathbf{r}(x)$. If $[g, b] : x \rightarrow y$ is a morphism in $\mathcal{X}/_p \mathcal{G}$ then we define $\Phi[g, b] := \mathbf{r}(b)\gamma$,

$$\mathbf{r}(x) \xrightarrow{\gamma} \mathbf{r}(xg) \xrightarrow{\mathbf{r}(b)} \mathbf{r}(y). \quad (4.12)$$

Using the naturality of γ one shows that $\Phi[g, b]$ is independent of the representative chosen for $[g, b]$. The fact that Φ sends composition of morphisms to the composition of the images follows by the higher coherence $(\gamma\beta)$ of Definition 3.22. Similarly, $\Phi(\text{id}) = \text{id}$ as a consequence of the higher coherence $(\gamma\varepsilon)$. Hence Φ is a functor that is a morphism of categories fibred in groupoids. It follows from the definition of Φ and the higher coherence $(\gamma\varepsilon)$ that $\Phi\mathbf{q} = \mathbf{r}$. Hence we can choose $\varphi := \text{id}_{\mathbf{r}}$, and $(\varphi\gamma)$ is satisfied because, by definition of Φ , we have $\Phi[g, \text{id}_{xg}] = \gamma$. This defines (Φ, φ) .

We now prove the uniqueness of (Φ, φ) . First, condition $(\psi\varphi\varrho)$ in the statement of the lemma defines ψ uniquely. We must verify that $\psi : \Phi \rightarrow \bar{\Phi}$ defined in this way is indeed a natural transformation. Let $[g, b] : x \rightarrow y$ be a morphism in $\mathcal{X}/_p\mathcal{G}$, and consider the following diagram:

$$\begin{array}{ccccc}
\Phi(x) & \xrightarrow{\Phi[g, b]} & \Phi(y) & & \\
\downarrow \varphi & \searrow \Phi[g, \text{id}] & \Phi(xg) & \swarrow \Phi(\mathbf{q}(b)) & \downarrow \varphi \\
\mathbf{r}(x) & \xrightarrow{\gamma} & \mathbf{r}(xg) & \xrightarrow{\mathbf{r}(b)} & \mathbf{r}(y) \\
\downarrow \varrho & & \downarrow \varrho & & \downarrow \varrho \\
\bar{\mathbf{r}}(x) & \xrightarrow{\bar{\gamma}} & \bar{\mathbf{r}}(xg) & \xrightarrow{\bar{\mathbf{r}}(b)} & \bar{\mathbf{r}}(y) \\
\uparrow \bar{\varphi} & \nwarrow \bar{\Phi}[g, \text{id}] & \bar{\Phi}(xg) & \searrow \bar{\Phi}(\mathbf{q}(b)) & \uparrow \bar{\varphi} \\
\bar{\Phi}(x) & \xrightarrow{\bar{\Phi}[g, b]} & \bar{\Phi}(y) & &
\end{array}$$

Since $\psi = (\bar{\varphi})^{-1}\varrho\varphi$, we have to prove that the outer rectangle commutes. Commutativity of the right trapezia follows from naturality of φ (the upper trapezium) and of $\bar{\varphi}$ (the lower one). Commutativity of the left trapezia follows from condition $(\varphi\gamma)$ applied to φ (the upper one) and to $\bar{\varphi}$ (the lower one). Commutativity of the upper and bottom triangles follows from $[g, b] = \mathbf{q}(b)[g, \text{id}]$, which in turn follows from the higher coherence $(xg1)$ of **(a4)**, Definition 3.15. The left middle square commutes by condition $(\varrho\gamma)$, and the right middle one by naturality of ϱ . This completes the proof of the uniqueness of (Φ, φ) . \square

The universal property described in Prop. 4.9 still holds when $\mathcal{X}/_p\mathcal{G}$ is replaced by its stackification \mathcal{X}/\mathcal{G} , as long as \mathcal{S} is a stack.

Corollary 4.10 *Suppose that a cfg-groupoid \mathcal{G} acts on $\mathcal{X} \in \text{Obj}(\text{CFG}_{\mathcal{C}})$ on the fibers of a morphism $\mathbf{r} : \mathcal{X} \rightarrow \mathcal{S}$ such that \mathcal{S} is a stack. Then there exists a pair (Φ, φ) , where $\Phi : \mathcal{X}/\mathcal{G} \rightarrow \mathcal{S}$ and $\varphi : \Phi\mathbf{q} \xrightarrow{\sim} \mathbf{r}$, with $\mathbf{q} : \mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}$ defined in (4.10), satisfying the properties described in (i) and (ii) of the previous proposition⁵.*

⁵To formulate properties (i) and (ii) of Prop. 4.9 in the present context, we note that in diagram $(\varphi\gamma)$ one has to substitute $\Phi(x)$ and $\Phi(xg)$ with $\Phi(\mathbf{q}(x))$ and $\Phi(\mathbf{q}(xg))$ respectively, and $\Phi[g, \text{id}_{xg}]$ by $\Phi(\gamma)$, where γ is the 2-isomorphism that makes \mathcal{G} act on the fibers of \mathbf{q} . Moreover, in the diagram $(\psi\varphi\varrho)$, one has to substitute $\Phi(x)$ and $\bar{\Phi}(x)$ by $\Phi(\mathbf{q}(x))$ and $\bar{\Phi}(\mathbf{q}(x))$, respectively.

PROOF: The proof follows from Prop. 4.9 and the properties of stackification, see Prop. 2.2 (one has to replace γ_0 by the 2-isomorphism γ that makes \mathcal{G} act on the fibers of $\mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}$, as in Remark 4.4). \square

Corollary 4.11 *If \mathcal{X} is a principal \mathcal{G} -bundle over a differentiable stack \mathcal{S} , then \mathcal{S} is canonically isomorphic to \mathcal{X}/\mathcal{G} .*

PROOF: Let $r : \mathcal{X} \rightarrow \mathcal{S}$ be the projection map of the principal bundle (see Def. 3.24), so that \mathcal{G} acts on its fibres. As mentioned in the previous corollary, there is an induced morphism $\mathcal{X}/\mathcal{G} \rightarrow \mathcal{S}$; we take it as the morphism induced by the morphism $\Phi : \mathcal{X}/_p \mathcal{G} \rightarrow \mathcal{S}$ constructed in the proof of Prop. 4.9. We must show that $\mathcal{X}/\mathcal{G} \rightarrow \mathcal{S}$ is an isomorphism.

Note that the fact that the map $\mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ in (3.10) is an isomorphism (condition 3. in Def. 3.24) implies that the \mathcal{G} -action on \mathcal{X} is 1-free (indeed, since the natural map $\mathcal{X} \times_{\mathcal{S}} \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is faithful, so is the action-projection map, cf. (4.14) below, which implies 1-freeness), so $\mathcal{X}/_p \mathcal{G}$ is a prestack (Prop. 4.8). Hence, by Prop. 2.2 (iv), it is enough to show that Φ is a monomorphism and an epimorphism.

The fact that Φ is an epimorphism follows directly from the fact that r is. We will show that Φ is a monomorphism, i.e., that for any manifold U the fiber of Φ at U , i.e., the functor $\Phi_U : (\mathcal{X}/_p \mathcal{G})_U \rightarrow \mathcal{S}_U$, is fully faithful. In what follows, all the objects and morphisms are understood to be over U ; in particular, all morphisms are isomorphisms.

Let $x, y \in \text{Obj}(\mathcal{X}/_p \mathcal{G}) = \text{Obj}(\mathcal{X})$, and let $a : r(x) \rightarrow r(y)$ be a morphism in \mathcal{S} (recall that $\Phi(x) = r(x)$ and $\Phi(y) = r(y)$). We have to show that there exists a unique morphism $[g, b] : x \rightarrow y$ in $\mathcal{X}/_p \mathcal{G}$ such that $\Phi[g, b] = a$.

Existence of $[g, b]$: The triple (x, a, y) is an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{X}$. Since we assume that the canonical morphism $\mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ is an isomorphism, there exists $(\bar{x}, g) \in \mathcal{X} \times_M \mathcal{G}$ and an isomorphism in $\mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ between $(\bar{x}, \bar{\gamma}, \bar{x}g)$ and (x, a, y) , where $\bar{\gamma}$ is induced by the isomorphism that makes \mathcal{G} act on the fibres of r ; in other words, we have isomorphisms $c : \bar{x} \rightarrow x$ and $b' : \bar{x}g \rightarrow y$ in \mathcal{X} such that the square

$$\begin{array}{ccc} r(\bar{x}) & \xrightarrow{r(c)} & r(x) \\ \bar{\gamma} \downarrow & & \downarrow a \\ r(\bar{x}g) & \xrightarrow{r(b')} & r(y) \end{array}$$

commutes. Defining $b := b' \circ (c \cdot \text{id})^{-1} : xg \rightarrow y$, we conclude that $\Phi[g, b] = a$ (cf. (4.12)).

Uniqueness of $[g, b]$: Let $[\bar{g}, \bar{b}] : x \rightarrow y$ be another morphism such that $\Phi[\bar{g}, \bar{b}] = a$. Then $(\text{id}, \bar{b}^{-1}b) : (x, \gamma, xg) \rightarrow (x, \bar{\gamma}, x\bar{g})$ is a morphism in $\mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ between the images of $(x, g), (x, \bar{g}) \in \text{Obj}(\mathcal{X} \times_M \mathcal{G})$; notice that this holds since $r(\bar{b})\bar{\gamma} = r(b)\gamma$, as both compositions are equal to a . Since $\mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ is an isomorphism, there exists a morphism $(c, j) : (x, g) \rightarrow (x, \bar{g})$ in $\mathcal{X} \times_M \mathcal{G}$ whose image in $\mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ is $(\text{id}, \bar{b}^{-1}b)$. In particular, $c = \text{id}$ and $(\text{id} \cdot j) = \bar{b}^{-1}b$. We conclude that $[g, b] = [\bar{g}, \bar{b}]$. \square

As a consequence of the last corollary, we note that any principal \mathcal{G} -bundle \mathcal{X} over a differentiable stack \mathcal{Y} , with projection $q_{\mathcal{Y}} : \mathcal{X} \rightarrow \mathcal{Y}$, satisfies the same universal property as the quotient $q : \mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}$ (cf. Corollary 4.10):

Corollary 4.12 *Let \mathcal{X} be a principal \mathcal{G} -bundle with projection $\mathbf{q}_{\mathcal{Y}} : \mathcal{X} \rightarrow \mathcal{Y}$. Suppose that \mathcal{G} acts on \mathcal{X} on the fibers of a morphism $\mathbf{r} : \mathcal{X} \rightarrow \mathcal{S}$ via γ , where \mathcal{S} is a differentiable stack. Then:*

- (i) *There exists a pair (Φ, φ) , where $\Phi : \mathcal{Y} \rightarrow \mathcal{S}$ and $\varphi : \Phi \mathbf{q}_{\mathcal{Y}} \xrightarrow{\sim} \mathbf{r}$ are as in (i) of Prop. 4.9 (suitably adapted to the present context, similarly to the footnote of Cor. 4.10).*
- (ii) *The pair (Φ, φ) is unique in the sense of (ii) of the same proposition.*

4.4 Equivariant maps and (pre)quotients

This section discusses the relation between equivariant maps and maps of (pre)quotients.

Consider a cfg-groupoid \mathcal{G} acting on $\mathcal{X}_i \in \text{Obj}(\text{CFG}_{\mathcal{C}})$, $i = 1, 2$, and let $F : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ be a \mathcal{G} -equivariant morphism, with associated equivariance 2-isomorphism δ (Def. 3.20). Let γ_0^i be the 2-isomorphism associated with the \mathcal{G} -action on the fibers of $\mathbf{q}_i : \mathcal{X}_i \rightarrow \mathcal{X}_i/p \mathcal{G}$, as in Remark 4.4.

The next proposition makes precise the fact that F induces a morphism $\mathcal{X}_1/p \mathcal{G} \rightarrow \mathcal{X}_2/p \mathcal{G}$.

Proposition 4.13 *The following holds:*

- (i) *There is a morphism $\Phi : \mathcal{X}_1/p \mathcal{G} \rightarrow \mathcal{X}_2/p \mathcal{G}$ and a 2-isomorphism $\varphi : \Phi \mathbf{q}_1 \xrightarrow{\sim} \mathbf{q}_2 F$ satisfying the following higher coherence condition: for any $(x, g) \in \mathcal{X}_1 \times_M \mathcal{G}$, the diagram*

$$\begin{array}{ccccc}
 (\varphi \gamma \delta) : & \Phi \mathbf{q}_1(x) & \xrightarrow{\Phi(\gamma_0^1)} & \Phi \mathbf{q}_1(xg) & \\
 & \varphi \downarrow & & \downarrow \varphi & \\
 & \mathbf{q}_2 F(x) & \xrightarrow{\gamma_0^2} \mathbf{q}_2(F(x)g) & \xrightarrow{\mathbf{q}_2(\delta)} \mathbf{q}_2 F(xg) &
 \end{array}$$

commutes.

- (ii) *Let $F, \overline{F} : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ be \mathcal{G} -equivariant morphisms, with associated equivariance 2-isomorphisms δ and $\bar{\delta}$, respectively, and let (Φ, φ) and $(\overline{\Phi}, \bar{\varphi})$ be corresponding pairs as in (i). Then, for any $\varsigma : F \xrightarrow{\sim} \overline{F}$ such that, for all $(x, g) \in \mathcal{X}_1 \times_M \mathcal{G}$, the diagram*

$$\begin{array}{ccc}
 (\varsigma \delta) : & F(x)g & \xrightarrow{\delta} F(xg) \\
 \varsigma \cdot \text{id} \downarrow & & \downarrow \varsigma \\
 & \overline{F}(x)g & \xrightarrow{\bar{\delta}} \overline{F}(xg)
 \end{array}$$

commutes, there exists a unique $\psi : \Phi \xrightarrow{\sim} \overline{\Phi}$ such that, for all $x \in \mathcal{X}_1$, the following diagram commutes:

$$\begin{array}{ccc}
 (\psi \varsigma) : & \Phi \mathbf{q}_1(x) & \xrightarrow{\psi} \overline{\Phi} \mathbf{q}_1(x) \\
 \varphi \downarrow & & \downarrow \bar{\varphi} \\
 & \mathbf{q}_2 F(x) & \xrightarrow{\mathbf{q}_2(\varsigma)} \mathbf{q}_2 \overline{F}(x).
 \end{array}$$

PROOF:

To prove (i), let us define $\tilde{\gamma}_2 := (\text{id}_{\mathbf{q}_2} \circ \delta) * (\gamma_0^2 \circ \text{id}_{F \times \text{id}})$; more explicitly, for $(x, g) \in \mathcal{X}_1 \times_M \mathcal{G}$ we have $\tilde{\gamma}_2 := \tilde{\gamma}_2(x, g) = \mathbf{q}_2(\delta) \circ \gamma_0^2(F(x), g)$,

$$\tilde{\gamma}_2 : \mathbf{q}_2(F(x)) \xrightarrow{\gamma_0^2} \mathbf{q}_2(F(x)g) \xrightarrow{\mathbf{q}_2(\delta)} \mathbf{q}_2(F(xg)).$$

One may check that $\tilde{\gamma}_2 : \mathbf{q}_2 F \text{pr}_{\mathcal{X}_1} \rightarrow \mathbf{q}_2 F \mathbf{A}_1$ makes \mathcal{G} act on the fibers of $\mathbf{q}_2 F$, i.e., that $\tilde{\gamma}_2$ satisfies the higher coherences described in Definition 3.22. Indeed, one directly checks that condition $(\gamma\beta)$ for $\tilde{\gamma}_2$ follows from the same condition for γ_0^2 and condition $(\delta\beta_1\beta_2)$ of Definition 3.20. Moreover, condition $(\gamma\varepsilon)$ for $\tilde{\gamma}_2$ follows from the same condition for γ_0^2 and condition $(\delta\varepsilon_1\varepsilon_2)$.

Next, we apply the universal property of the quotient map $\mathcal{X}_1 \rightarrow \mathcal{X}_1/p \mathcal{G}$ to the morphism $\mathbf{q}_2 F$ using Prop. 4.9. It follows that there exists (Φ, φ) , where $\Phi : \mathcal{X}_1/p \mathcal{G} \rightarrow \mathcal{X}_2/p \mathcal{G}$ and $\varphi : \Phi_{\mathbf{q}_1} \rightarrow \mathbf{q}_2 F$, satisfying the higher coherence given in part (i) of Prop. 4.9. This condition is equivalent to condition $(\varphi\gamma\delta)$, so (i) follows.

The assertion in (ii) is a restatement of part (ii) of Prop. 4.9 (with $\varrho = \text{id}_{\mathbf{q}_2} \circ \varsigma : \mathbf{q}_2 F \rightarrow \mathbf{q}_2 \bar{F}$). \square

The construction taking a \mathcal{G} -equivariant morphism $F : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ to a morphism $\Phi : \mathcal{X}_1/p \mathcal{G} \rightarrow \mathcal{X}_2/p \mathcal{G}$ preserves compositions and takes the identity to the identity. We state the precise result about compositions, leaving the proof to the reader.

Proposition 4.14 *Let \mathcal{G} be a cfg-groupoid acting on $\mathcal{X}_i \in \text{Obj}(\text{CFG}_{\mathcal{C}})$, for $i = 1, 2, 3$, and let $F_1 : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ and $F_2 : \mathcal{X}_2 \rightarrow \mathcal{X}_3$ be \mathcal{G} -equivariant morphisms, with associated equivariance 2-isomorphisms δ_1 and δ_2 . Let (Φ_1, φ_1) and (Φ_2, φ_2) be pairs as in Proposition 4.13(i) corresponding to F_1 and F_2 , respectively. Then the pair*

$$(\Phi_2 \Phi_1, (\varphi_2 \circ \text{id}_{F_1}) * (\text{id}_{\Phi_2} \circ \varphi_1))$$

corresponds to $F_2 F_1$.

Remark 4.15

- (a) The above functoriality property of the induced map implies that if F is an isomorphism then so is Φ .
- (b) It is immediate (using Corollary 4.10) that Propositions 4.13, 4.14, and part (a) above still work if we consider the quotient \mathcal{X}/\mathcal{G} instead of the prequotient.

We finally observe that passing from equivariant maps to maps of quotients preserves the monomorphism and epimorphism conditions.

Proposition 4.16 *Let $F : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ be a \mathcal{G} -equivariant map between cfg's, where \mathcal{G} is a cfg-groupoid. Let $\Phi : \mathcal{X}_1/p \mathcal{G} \rightarrow \mathcal{X}_2/p \mathcal{G}$ be the induced map on the prequotients. If F is a monomorphism (resp. an epimorphism) then so is Φ .*

PROOF: Since the canonical projections $\mathcal{X}_i \rightarrow \mathcal{X}_i/p \mathcal{G}$ are the identity on objects, it follows immediately that if F is an epimorphism then so is Φ . For the remaining part of the proof we fix a manifold U , and we assume that the morphisms considered are over id_U .

We first prove that if F_U is faithful, then so is Φ_U . Take two morphisms $[g, b], [\bar{g}, \bar{b}] : x \rightarrow x'$ in $\mathcal{X}_1/p \mathcal{G}$, and assume that $\Phi[g, b] = \Phi[\bar{g}, \bar{b}]$. Recall that $b : x \cdot g \rightarrow x'$ is a morphism in \mathcal{X}_1 .

Since $\Phi[g, b] = [g, F(b)]$ (after the identification of $F(x) \cdot g$ with $F(x' \cdot g)$ by the equivariance of F) then there exists an isomorphism $j : g \rightarrow \bar{g}$ such that $F(\bar{b}) \circ (\text{id}_{F(x)} \cdot j) = F(b)$. Then, again using the equivariance of F , $F(\bar{b} \circ (\text{id}_x \cdot j)) = F(b)$. Since F_U is faithful then $\bar{b} \circ (\text{id}_x \cdot j) = b$, hence $[g, b] = [\bar{g}, \bar{b}]$.

We now verify that if F_U is full, then so is Φ_U . Take objects $x, x' \in \text{Obj}(\mathcal{X}_1/p \mathcal{G}) = \text{Obj}(\mathcal{X}_1)$ and a morphism $[g, c] : F(x) \rightarrow F(x')$ in $\mathcal{X}_2/p \mathcal{G}$. By the equivariance of F we can interpret $c : F(x \cdot g) \rightarrow F(x')$ as a morphism between the images via F of objects in \mathcal{X}_1 . Since F_U is full, there exists $b : x \cdot g \rightarrow x'$ in \mathcal{X}_1 such that $F(b) = c$. We conclude $\Phi[g, b] = [g, c]$, and we are done. \square

4.5 Prequotients and the action-projection map

Let us consider a cfg-groupoid $\mathcal{G} \rightrightarrows M$ acting on $\mathcal{X} \in \text{Obj}(\text{CFG}_{\mathcal{C}})$ (on the right), and let $\Delta : \mathcal{X} \times_{a, M, t} \mathcal{G} \rightarrow \mathcal{X} \times \mathcal{X}$, $\Delta(x, g) = (x, xg)$, be the associated action-projection map. Using the projection $q : \mathcal{X} \rightarrow \mathcal{X}/p \mathcal{G}$, we see that Δ induces a morphism (cf. (3.10))

$$\widehat{\Delta} : \mathcal{X} \times_{a, M, t} \mathcal{G} \rightarrow \mathcal{X} \times_{\mathcal{X}/p \mathcal{G}} \mathcal{X}, \quad (4.13)$$

such that the diagram

$$\begin{array}{ccc} \mathcal{X} \times_M \mathcal{G} & \xrightarrow{\widehat{\Delta}} & \mathcal{X} \times_{\mathcal{X}/p \mathcal{G}} \mathcal{X} \\ & \searrow \Delta & \downarrow \\ & & \mathcal{X} \times \mathcal{X} \end{array} \quad (4.14)$$

2-commutes.

Proposition 4.17 *The following holds:*

- (a) $\widehat{\Delta}$ is full and essentially surjective.
- (b) If Δ is faithful, then so is $\widehat{\Delta}$.

Hence, if Δ is faithful then $\widehat{\Delta}$ is an isomorphism.

PROOF: The claim in (b) follows from the factorization (4.14).

Let us prove (a). To verify that $\widehat{\Delta}$ is full, let us consider objects

$$(x, g), (\bar{x}, \bar{g}) \in \mathcal{X} \times_M \mathcal{G},$$

and a morphism between the images of these objects in $\mathcal{X} \times_{\mathcal{X}/p \mathcal{G}} \mathcal{X}$:

$$(c_1, c_2) : (x, [g, \text{id}_{x \cdot g}], x \cdot g) \rightarrow (\bar{x}, [\bar{g}, \text{id}_{\bar{x} \cdot \bar{g}}], \bar{x} \cdot \bar{g}).$$

We have to prove the existence of $(b, j) : (x, g) \rightarrow (\bar{x}, \bar{g})$ in $\mathcal{X} \times_M \mathcal{G}$ such that $(b, b \cdot j) = (c_1, c_2)$. Since necessarily $b = c_1$ then, given (c_1, c_2) , we are looking for a morphism in \mathcal{G} ,

$$j : g \rightarrow \bar{g},$$

such that $t(j) = a(c_1)$ and $c_2 = c_1 \cdot j$.

In order to proceed we observe that $c_1 : x \rightarrow \bar{x}$ and $c_2 : x \cdot g \rightarrow \bar{x} \cdot \bar{g}$ are morphisms in \mathcal{X} with

$$\pi_{\mathcal{X}}(c_1) = \pi_{\mathcal{X}}(c_2) =: \mu,$$

and that the square

$$\begin{array}{ccc} x & \xrightarrow{[g, \text{id}_{x \cdot g}]} & x \cdot g \\ \downarrow [1_{\mathbf{a}(x)}, c_1 \circ \varepsilon(x)] & & \downarrow [1_{\mathbf{a}(x \cdot g)}, c_2 \circ \varepsilon(x \cdot g)] \\ \bar{x} & \xrightarrow{[\bar{g}, \text{id}_{\bar{x} \cdot \bar{g}}]} & \bar{x} \cdot \bar{g} \end{array}$$

commutes in $\mathcal{X}/_p \mathcal{G}$. This implies that there exists

$$j' : g \cdot 1_{\mathbf{a}(x \cdot g)} \xrightarrow{\sim} 1_{\mathbf{a}(x)} \cdot \mu^* \bar{g}$$

such that $\mathbf{t}(j') = \text{id}_{\mathbf{a}(x)}$ and the diagram

$$\begin{array}{ccc} x \cdot (g \cdot 1_{\mathbf{a}(x \cdot g)}) & \xrightarrow{\beta} & (x \cdot g) \cdot 1_{\mathbf{a}(x \cdot g)} \\ \text{id} \cdot j' \downarrow & & \downarrow \varepsilon \\ x \cdot (1_{\mathbf{a}(x)} \cdot \mu^* \bar{g}) & & x \cdot g \xrightarrow{c_2} \bar{x} \cdot \bar{g} \\ \beta \downarrow & \nearrow c_1 \cdot \mu_{\bar{g}} & \\ (x \cdot 1_{\mathbf{a}(x)}) \cdot \mu^* \bar{g} & \xrightarrow{\varepsilon \cdot \text{id}} & x \cdot \mu^* \bar{g} \end{array} \quad (4.15)$$

commutes, where $\mu_{\bar{g}} : \mu^* \bar{g} \rightarrow \bar{g}$ is a cartesian arrow in \mathcal{G} over μ . Since $1_{\mathbf{a}(x \cdot g)} = 1_{\mathbf{s}(g)}$ and $1_{\mathbf{a}(x)} = 1_{\mathbf{t}(\mu^* \bar{g})}$, we can consider the following diagram of isomorphisms in \mathcal{G} :

$$g \xrightarrow{(\rho g)^{-1}} g \cdot 1_{\mathbf{a}(x \cdot g)} \xrightarrow{j'} 1_{\mathbf{a}(x)} \cdot \mu^* \bar{g} \xrightarrow{\lambda(\mu^* \bar{g})} \mu^* \bar{g} \xrightarrow{\mu_{\bar{g}}} \bar{g},$$

and define j to be the above composition:

$$j := \mu_{\bar{g}} \circ \lambda(\mu^* \bar{g}) \circ j' \circ (\rho g)^{-1}.$$

Since $\mathbf{t}(\mu_{\bar{g}}) = \mathbf{a}(c_1)$, $\mathbf{t}(\lambda(\mu^* \bar{g})) = \mathbf{t}(j') = \mathbf{t}(\rho g)^{-1} = \text{id}_{\mathbf{a}(x)}$, then $\mathbf{t}(j) = \mathbf{a}(c_1)$. We are left with checking that $c_2 = c_1 \cdot j$, and this follows by completing the diagram in (4.15) to

$$\begin{array}{ccc} x \cdot (g \cdot 1_{\mathbf{a}(x \cdot g)}) & \xrightarrow{\beta} & (x \cdot g) \cdot 1_{\mathbf{a}(x \cdot g)} \\ \text{id} \cdot j' \downarrow & \searrow \text{id} \cdot \rho & \downarrow \varepsilon \\ x \cdot (1_{\mathbf{a}(x)} \cdot \mu^* \bar{g}) & & x \cdot g \xrightarrow{c_2} \bar{x} \cdot \bar{g} \\ \beta \downarrow & \searrow \text{id} \cdot \lambda & \nearrow c_1 \cdot \mu_{\bar{g}} \\ (x \cdot 1_{\mathbf{a}(x)}) \cdot \mu^* \bar{g} & \xrightarrow{\varepsilon \cdot \text{id}} & x \cdot \mu^* \bar{g} \end{array}$$

noticing that the two triangles commute by the higher coherences $(xg1)$ and $(x1g)$ in **(a4)**, all the arrows but possibly c_2 and $(c_1 \cdot \mu_{\bar{g}})$ are isomorphisms, and $c_2 = (c_1 \cdot \mu_{\bar{g}}) \circ (\text{id} \cdot \lambda) \circ (\text{id} \cdot j') \circ (\text{id} \cdot \rho^{-1}) = c_1 \cdot j$. This completes the proof that $\hat{\Delta}$ is full.

It remains to show that $\widehat{\Delta}$ is essentially surjective. Given an object

$$(x_1, [g, b], x_2) \in \text{Obj}(\mathcal{X} \times_{\mathcal{X}/p} \mathcal{G}),$$

we have to show that there exists an object

$$(x, h) \in \text{Obj}(\mathcal{X} \times_M \mathcal{G})$$

and an isomorphism

$$(c_1, c_2) : (x, [h, \text{id}_{x \cdot h}], x \cdot h) \xrightarrow{\sim} (x_1, [g, b], x_2) \quad \text{in } \mathcal{X} \times_{\mathcal{X}/p} \mathcal{G}.$$

Let

$$x := x_1, \quad h := g,$$

and

$$c_1 := \text{id}_{x_1} : x_1 \longrightarrow x_1 \quad c_2 := b : x_1 \cdot g \longrightarrow x_2$$

in \mathcal{X} . The first condition to be verified is that the diagram

$$\begin{array}{ccc} x_1 & \xrightarrow{[g, \text{id}_{x_1 \cdot g}]} & x_1 \cdot g \\ \downarrow [\mathbf{1}_{\mathbf{a}(x_1)}, \varepsilon(x_1)] & & \downarrow [\mathbf{1}_{\mathbf{a}(x_1 \cdot g)}, b \circ \varepsilon(x_1 \cdot g)] \\ x_1 & \xrightarrow{[g, b]} & x_2 \end{array}$$

commutes in $\mathcal{X} \times_{\mathcal{X}/p} \mathcal{G}$. To do that, we have to show that there exists an isomorphism in \mathcal{G} ,

$$j : \mathbf{1}_{\mathbf{a}(x_1)} \cdot g \xrightarrow{\sim} g \cdot \mathbf{1}_{\mathbf{a}(x_1 \cdot g)},$$

such that $\mathbf{t}(j) = \text{id}_{\mathbf{a}(x_1)}$ and

$$\begin{array}{ccc} x_1 \cdot (\mathbf{1}_{\mathbf{a}(x_1)} \cdot g) & \xrightarrow{\beta} & (x_1 \cdot \mathbf{1}_{\mathbf{a}(x_1)}) \cdot g \\ \downarrow \text{id} \cdot j & & \downarrow \varepsilon \cdot \text{id} \\ & & x_1 \cdot g \xrightarrow{b} x_2 \\ & & \uparrow \varepsilon \\ x_1 \cdot (g \cdot \mathbf{1}_{\mathbf{a}(x_1 \cdot g)}) & \xrightarrow{\beta} & (x_1 \cdot g) \cdot \mathbf{1}_{\mathbf{a}(x_1 \cdot g)} \end{array} \quad (4.16)$$

commutes. Since $\mathbf{1}_{\mathbf{a}(x_1)} = \mathbf{1}_{\mathbf{t}(g)}$ and $\mathbf{1}_{\mathbf{a}(x_1 \cdot g)} = \mathbf{1}_{\mathbf{s}(g)}$, we can define

$$j := (\rho g)^{-1} \cdot \lambda g : \mathbf{1}_{\mathbf{a}(x_1)} \cdot g \xrightarrow{\sim} g \xrightarrow{\sim} g \cdot \mathbf{1}_{\mathbf{a}(x_1 \cdot g)}.$$

Since $\rho(g)$ and $\lambda(g)$ are sent to the identity by $\pi_{\mathcal{G}}$, it follows that $\mathbf{t}(j) = \text{id}_{\mathbf{a}(x_1)}$. The commutativity of the diagram (4.16) is shown by the diagram

$$\begin{array}{ccc} x_1 \cdot (\mathbf{1}_{\mathbf{a}(x_1)} \cdot g) & \xrightarrow{\beta} & (x_1 \cdot \mathbf{1}_{\mathbf{a}(x_1)}) \cdot g \\ \downarrow \text{id} \cdot j & \searrow \text{id} \cdot \lambda & \downarrow \varepsilon \cdot \text{id} \\ & & x_1 \cdot g \xrightarrow{b} x_2 \\ & \nearrow \text{id} \cdot \rho & \uparrow \varepsilon \\ x_1 \cdot (g \cdot \mathbf{1}_{\mathbf{a}(x_1 \cdot g)}) & \xrightarrow{\beta} & (x_1 \cdot g) \cdot \mathbf{1}_{\mathbf{a}(x_1 \cdot g)} \end{array}$$

noticing that the upper and lower triangles commute by the higher coherences $(x1g)$ and $(xg1)$ of **(a4)** in Def. 3.15. This proves that (c_1, c_2) is a morphism in $\mathcal{X} \times_{\mathcal{X}/\mathcal{G}} \mathcal{X}$, hence an isomorphism since c_1 and c_2 are over an identity map in \mathcal{C} . This concludes the proof that $\hat{\Delta}$ is essentially surjective. \square

5 Principal actions of stacky Lie groupoids

This section concerns actions that give rise to stacky principal bundles.

Definition 5.1 *If \mathcal{G} is a stacky Lie groupoid and \mathcal{X} is a differentiable stack, we call a \mathcal{G} -action on \mathcal{X} **principal** if \mathcal{X}/\mathcal{G} is a differentiable stack and \mathcal{X} is a principal \mathcal{G} -bundle over \mathcal{X}/\mathcal{G} .*

Since we know that \mathcal{G} always acts on \mathcal{X} on the fibres of the quotient map $q : \mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}$ (see Remark 4.4), the previous definition amounts to showing that \mathcal{X}/\mathcal{G} is a differentiable stack such that q is an epimorphism and a submersion, and that the natural map (3.10),

$$\mathcal{X} \times_M \mathcal{G} \longrightarrow \mathcal{X} \times_{\mathcal{X}/\mathcal{G}} \mathcal{X}, \quad (5.1)$$

is an isomorphism.

We will provide in this section a simple characterization of principal actions, which could be thought of as parallel to the characterization of principal actions in the smooth category by free and proper actions.

5.1 The main theorem: characterization of principal actions

Recall the notion of weak representability from Def. 2.6. The following is our main result.

Theorem 5.2 *Let $\mathcal{G} \rightrightarrows M$ be a stacky Lie groupoid acting (on the right) on a differentiable stack \mathcal{X} along $a : \mathcal{X} \rightarrow M$.*

- (a) *If the action-projection map $\Delta : \mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X} \times \mathcal{X}$ is weakly representable, then the \mathcal{G} -action on \mathcal{X} is principal.*
- (b) *Conversely, suppose \mathcal{X} is a principal \mathcal{G} -bundle over a differentiable stack \mathcal{S} . Then the action-projection map of the action is weakly representable, and \mathcal{S} is canonically isomorphic to \mathcal{X}/\mathcal{G} .*

PROOF:

Part (a): Notice that by Prop. 2.8 we know that the action-projection map Δ is automatically faithful, so the quotient map $q : \mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}$ is an epimorphism as a consequence of Propositions 4.8 and 4.6(a), and the map (5.1) is an isomorphism as a result of Propositions 4.17 and 4.6(b). So, to conclude part (a), it remains to show that \mathcal{X}/\mathcal{G} is a differentiable stack so that $q : \mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}$ is a submersion.

Let $G \rightrightarrows G_0$ and $X \rightrightarrows X_0$ be Lie groupoids presenting \mathcal{G} and \mathcal{X} . We will prove that the morphism

$$X_0 \rightarrow \mathcal{X}/\mathcal{G}, \quad (5.2)$$

given by the composition $X_0 \rightarrow \mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}$, is representable. Assuming this fact, and recalling that the quotient map $q : \mathcal{X} \rightarrow \mathcal{X}/\mathcal{G}$ is an epimorphism, we see that (5.2) is an epimorphism, so

it is an atlas for \mathcal{X}/\mathcal{G} , i.e., this quotient is differentiable. Moreover, Prop. 2.10, part (b), implies that \mathbf{q} is a submersion.

So we are left with verifying the representability of the morphism (5.2), and according to Lemma A.1 it suffices to prove that:

1. The fibred product $X_0 \times_{\mathcal{X}/\mathcal{G}} X_0$ is represented by a manifold, say V .
2. The induced map $V \rightarrow X_0$ is a submersion. (There are two such induced maps; it is enough to prove the statement for one of them.)

The proofs of conditions 1. and 2. above follow from Proposition 5.5 and Lemmas 5.7 and 5.8, presented in Sections 5.2 and 5.3 below. We outline the steps.

To prove 1., let E be the (total space of the) Hilsum-Skandalis bibundle associated with the action map $\mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X}$. Recall that the fact that the action-projection map is faithful implies that the action is 1-free. So we can use Proposition 5.5 below to conclude that E carries a left action of the Lie groupoid $G \rightrightarrows G_0$ in such a way that the quotient stack $[G \backslash E]$ is described as a fibred product:

$$\begin{array}{ccc} [G \backslash E] & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & \mathcal{X}/\mathcal{G}. \end{array}$$

The conclusion in 1. follows if we show that $[G \backslash E]$ is representable.

Lemma 5.7 shows that we have a 2-cartesian diagram

$$\begin{array}{ccc} [X \times_M G \backslash X \times_{X_0} E] & \longrightarrow & X_0 \times X_0 \\ \downarrow & & \downarrow \\ \mathcal{X} \times_M \mathcal{G} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}, \end{array}$$

while Lemma 5.8 gives an isomorphism

$$[X \times_M G \backslash X \times_{X_0} E] \xrightarrow{\sim} [G \backslash E].$$

From the assumption that Δ is weakly representable, it follows that $[X \times_M G \backslash X \times_{X_0} E]$ is representable, and hence so is $[G \backslash E]$. This concludes the proof of 1.

In order to prove 2., we will see in Section 5.2 that we have a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & [G \backslash E] \\ & \searrow & \downarrow \\ & & X_0 \end{array}$$

in which the map $E \rightarrow X_0$ is a submersion, see Remark 5.4. Since the horizontal map is surjective (as a map between manifolds), it follows that the vertical map is a submersion, as desired.

Part (b): In order to verify that the action-projection map $\Delta : \mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X} \times \mathcal{X}$ is weakly representable, let U be a manifold, and let $U \rightarrow \mathcal{X}$ be a representable morphism (e.g. an atlas). We have a cartesian diagram

$$\begin{array}{ccc} U \times_S U & \longrightarrow & U \times U \\ \downarrow & & \downarrow \\ \mathcal{X} \times_S \mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X}. \end{array}$$

Since $\mathcal{X} \rightarrow \mathcal{S}$ is assumed to be a submersion, the composition $U \rightarrow \mathcal{S}$ is representable, so $Z := U \times_{\mathcal{S}} U$ is representable. Using that the map $\mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X} \times_{\mathcal{S}} \mathcal{X}$ (induced by Δ) is an isomorphism, we see that Z fits into a cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & U \times U \\ \downarrow & & \downarrow \\ \mathcal{X} \times_M \mathcal{G} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}, \end{array}$$

and we conclude that Δ is weakly representable.

The assertion that \mathcal{S} is canonically identified with \mathcal{X}/\mathcal{G} is proven in Cor. 4.11. \square

An important instance of Theorem 5.2 is when the stacky Lie groupoid $\mathcal{G} \rightrightarrows M$ is representable, i.e., isomorphic to an ordinary Lie groupoid. In this case the conditions in the theorem are automatically satisfied:

Corollary 5.3 *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and suppose that it acts on a differentiable stack \mathcal{X} . Then \mathcal{X}/\mathcal{G} is a differentiable stack and the \mathcal{G} -action on \mathcal{X} is principal over it.*

PROOF: We must check that the action-projection map is weakly representable. Note that by Lemmas 5.7 and 5.8 (cf. proof above), it suffices to show that $[G \setminus E]$ is representable, where E is the Hilsum-Skandalis bibundle presenting the action map $\mathcal{X} \times_{\mathcal{G}} \mathcal{G} \rightarrow \mathcal{X}$, and $G \rightrightarrows G_0$ presents \mathcal{G} . By the assumption on \mathcal{G} being representable, we may take $G = G_0 = \mathcal{G}$ (the trivial groupoid), in which case $[G \setminus E] = E$. \square

When the Lie groupoid in the previous corollary is a Lie group, this recovers [20, Thm. 0.2]. When both \mathcal{G} and \mathcal{X} are representable, we are in the situation of an ordinary Lie groupoid action of $G \rightrightarrows M$ on a manifold X , in which case \mathcal{X}/\mathcal{G} is the quotient stack $[X/G]$, and we recover the well-known fact that X is a principal G -bundle over it (see Example 3.27).

5.2 Proofs of lemmas

We now present the lemmas used to prove Theorem 5.2.

Let a stacky Lie groupoid $\mathcal{G} \rightrightarrows M$ act (on the right) on the differentiable stack \mathcal{X} . As in Section 5.1, let $G \rightrightarrows G_0$ and $X \rightrightarrows X_0$ be presentations of \mathcal{G} and \mathcal{X} , respectively. We identify \mathcal{G} with BG and \mathcal{X} with BX , and use the following notation: the action of BG on BX is along $\mathbf{p} : BX \rightarrow M$, and $\mathbf{s}, \mathbf{t} : BG \rightarrow M$ are the source and target maps. The compositions of these structure maps with the atlas maps $X_0 \rightarrow BX$ and $G_0 \rightarrow BG$ give rise to groupoid morphisms

$$\begin{array}{ccc} \begin{array}{ccc} X & \xrightarrow{\bar{p}} & M \\ s_X \downarrow & & \downarrow \\ X_0 & \xrightarrow{p} & M \end{array} & \begin{array}{ccc} G & \xrightarrow{\bar{s}} & M \\ s_G \downarrow & & \downarrow \\ G_0 & \xrightarrow{s} & M \end{array} & \begin{array}{ccc} G & \xrightarrow{\bar{t}} & M \\ s_G \downarrow & & \downarrow \\ G_0 & \xrightarrow{t} & M \end{array} \end{array} \quad (5.3)$$

where $M \rightrightarrows M$ is the trivial groupoid, and the following equalities hold:

$$\bar{p} = p \circ s_X = p \circ t_X, \quad \bar{s} = s \circ s_G = s \circ t_G, \quad \bar{t} = t \circ s_G = t \circ t_G.$$

The fibred product

$$BX \times_M BG := BX \times_{\mathbf{p}, M, \mathbf{t}} BG$$

is presented by the groupoid $X \times_{\bar{p}, M, \bar{t}} G \rightrightarrows X_0 \times_{p, M, t} G_0$ (the “1-categorical” fibred product, as in [34]), that we write more simply as

$$X \times_M G \rightrightarrows X_0 \times_M G_0.$$

The action morphism

$$BX \times_M BG \longrightarrow BX$$

is presented by a right principal bibundle that we denote by

$$\begin{array}{ccc} X \times_M G & & X \\ \Downarrow & & \Downarrow \\ X_0 \times_M G_0 & \xleftarrow{a} E \xrightarrow{b} & X_0. \end{array} \quad (5.4)$$

Let p_t and t_p denote the following natural maps:

$$\begin{array}{ccc} X_0 \times_M G_0 & \xrightarrow{p_t} & G_0 \\ t_p \downarrow & & \downarrow t \\ X_0 & \xrightarrow{p} & M. \end{array} \quad (5.5)$$

We will consider a G -action on E along the map $p_t a : E \rightarrow G_0$, given by

$$g \cdot e := (1_{t_p a(e)}, g) \cdot e, \quad (5.6)$$

for $g \in G$ and $e \in E$ with $s_G(g) = p_t a(e)$, where on the right-hand side we use the $(X \times_M G)$ -action on E .

Since G acts on the fibers of $b : E \rightarrow X_0$ and $t_p a : E \rightarrow X_0$, we have induced morphisms between stacks,

$$\tilde{b} : [G \backslash E] \longrightarrow X_0, \quad \widetilde{t_p a} : [G \backslash E] \longrightarrow X_0,$$

fitting into the following commutative triangles:

$$\begin{array}{ccc} E & \longrightarrow & [G \backslash E] \\ & \searrow b & \downarrow \tilde{b} \\ & & X_0 \end{array} \quad \begin{array}{ccc} E & \longrightarrow & [G \backslash E] \\ & \searrow t_p a & \downarrow \widetilde{t_p a} \\ & & X_0. \end{array}$$

More explicitly, the induced morphisms are defined as follows. Given an object (P, c) in $[G \backslash E]$, i.e., a left principal G -bundle P and a G -equivariant morphism $c : P \rightarrow E$,

$$\begin{array}{ccc} P & \xrightarrow{c} & E \\ \downarrow & & \\ U & & \end{array}$$

then $\tilde{b}(P, c) : U \rightarrow X_0$ is the unique morphism that makes the following diagram commute:

$$\begin{array}{ccc} P & \xrightarrow{c} & E \\ \downarrow & & \downarrow b \\ U & \longrightarrow & X_0. \end{array}$$

Given a morphism in $[G \backslash E]$,

$$\begin{array}{ccc} P & \longrightarrow & P' \\ \downarrow & & \downarrow \\ U & \xrightarrow{\mu} & U' \end{array}$$

its image under \tilde{b} is given by μ . The morphism $\widetilde{t_p a}$ is defined similarly.

Remark 5.4 Note that $t_p a : E \rightarrow X_0$ is a submersion since both a and t_p are (as a result of E being right principal and t_p being a base change of $t : G_0 \rightarrow M$, which is a submersion since $t : \mathcal{G} \rightarrow M$ is a submersion).

The next proposition is the first key result in the proof of part (a) of Theorem 5.2.

Proposition 5.5 *If the action of BG on BX is 1-free, then there is a canonical 2-isomorphism making the following square 2-cartesian:*

$$\begin{array}{ccc} [G \backslash E] & \xrightarrow{\tilde{b}} & X_0 \\ \widetilde{t_p a} \downarrow & & \downarrow \\ X_0 & \longrightarrow & BX/BG. \end{array}$$

Since the proof of this result is lengthy, we will present it separately in Section 5.3.

We now consider a (left) action of the Lie groupoid $X \times_M G \rightrightarrows X_0 \times_M G_0$ on

$$X \times_{X_0} E := X \times_{t_X, X_0, t_p a} E$$

along the map

$$X \times_{X_0} E \longrightarrow X_0 \times_M G_0, \quad (x, e) \mapsto a(e),$$

as follows:

$$(x', g) \cdot (x, e) = (x'x, (x', g)e).$$

The space $X \times_{X_0} E$ also carries a right action of $X \times X$, along the map (s_X, b) , by $(x, e) \cdot (x_1, x_2) = (xx_1, ex_2)$, making it into a bibundle

$$\begin{array}{ccc} X \times_M G & & X \times X \\ \Downarrow & & \Downarrow \\ X_0 \times_M G_0 & \longleftarrow X \times_{X_0} E \longrightarrow & X_0 \times X_0. \end{array} \quad (5.7)$$

Lemma 5.6 *The bibundle (5.7) defines a Hilsum-Skandalis map corresponding to the action-projection map $\Delta : \mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X} \times \mathcal{X}$.*

PROOF: The proof follows from the fact that the Hilsum-Skandalis bibundle corresponding to the projection $\mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X}$ is

$$\begin{array}{ccc} X \times_M G & & X \\ \Downarrow & & \Downarrow \\ X_0 \times_M G_0 & \longleftarrow G_0 \times_M X \longrightarrow & X_0, \end{array}$$

as described in Lemma 2.15. The result is now an application of Lemma 2.16. \square

Lemma 5.7 *There is a canonical 2-cartesian square*

$$\begin{array}{ccc} [X \times_M G \backslash X \times_{X_0} E] & \longrightarrow & X_0 \times X_0 \\ \downarrow & & \downarrow \\ \mathcal{X} \times_M \mathcal{G} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}. \end{array}$$

PROOF: The proof follows from Proposition 3.35 and Lemma 5.6. \square

Lemma 5.8 *There is an isomorphism*

$$[X \times_M G \backslash X \times_{X_0} E] \xrightarrow{\sim} [G \backslash E].$$

PROOF: Both quotient stacks are presented by the translation groupoids of the corresponding actions. The translation groupoid presenting $[X \times_M G \backslash X \times_{X_0} E]$ is

$$(X \times_M G) \times_{X_0 \times_M G_0} (X \times_{X_0} E) \rightrightarrows X \times_{X_0} E$$

where source and target maps are given, respectively, by

$$(x', g, x, e) \mapsto (x, e), \quad (x', g, x, e) \mapsto (x'x, (x', g)e)$$

and multiplication by

$$(x'_1, g_1, x_1, e_1) \cdot (x'_2, g_2, x_2, e_2) = (x'_1 x'_2, g_1 g_2, x_2, e_2).$$

The translation groupoid of $[G \backslash E]$ is $G \times_{G_0} E \rightrightarrows E$, with source and target maps given, respectively, by

$$(g, e) \mapsto e, \quad (g, e) \mapsto (1, g)e,$$

and multiplication

$$(g, e) \cdot (g', e') = (gg', e').$$

The assertion in the lemma follows from the fact that there is a weak equivalence (as in [34, Sec. 5.4], see also comments below Eq. (2.5)) between the two translation groupoids,

$$(X \times_M G) \times_{X_0 \times_M G_0} (X \times_{X_0} E) \rightarrow G \times_{G_0} E,$$

given on objects by $(x, e) \mapsto (x^{-1}, 1)e$ (using the action of $X \times_M G$ on E), and on arrows by $(x', g, x, e) \mapsto (g, (x^{-1}, 1)e)$. \square

5.3 Proof of Proposition 5.5

For the proof of Prop. 5.5, we keep the notation of Section 5.2.

Since we assume that the action of $\mathcal{G} = BG$ on $\mathcal{X} = BX$ is 1-free, it follows from Propositions 4.6(b) and 4.8 that

$$X_0 \times_{BX/BG} X_0 = X_0 \times_{BX/p_B G} X_0.$$

We will work with the latter stack, for which we have a more explicit description: objects are triples $(\hat{a}, [Q, \varphi], \hat{b})$, where $\hat{a}, \hat{b} : U \rightarrow X_0$, U is a manifold, and $[Q, \varphi] : \hat{a}^* X \rightarrow \hat{b}^* X$ is a morphism over U in the prequotient $BX/_p BG$ (as in (4.2)); recall, in particular, that Q is a principal G -bundle over U such that $\hat{a}^* X \cdot Q$ is defined ($\hat{a}^* X \cdot Q$ is the X -bundle defined by the action $BX \times_M BG \rightarrow BX$), and $\varphi : \hat{a}^* X \cdot Q \rightarrow \hat{b}^* X$ is an isomorphism of principal X -bundles over U . We keep the notation $\hat{a}^* X$ for the pullback by \hat{a} of X viewed as a principal X -bundle through multiplication on the right (similarly for $\hat{b}^* X$). Morphisms in $X_0 \times_{BX/_p BG} X_0$ are pairs

$$(\mu_a, \mu_b) : (\hat{a}, [Q, \varphi], \hat{b}) \rightarrow (\hat{a}_1, [Q_1, \varphi_1], \hat{b}_1)$$

where $\mu_a, \mu_b : U \rightarrow U_1$ are such that $\hat{a} = \hat{a}_1 \mu_a$, $\hat{b} = \hat{b}_1 \mu_b$ and the diagram

$$\begin{array}{ccc} \hat{a}^* X & \xrightarrow{[Q, \varphi]} & \hat{b}^* X \\ \downarrow & & \downarrow \\ \hat{a}_1^* X & \xrightarrow{[Q_1, \varphi_1]} & \hat{b}_1^* X \end{array}$$

commutes in $BX/_p BG$ (the vertical maps in the diagram are those naturally induced by μ_a and μ_b).

Consider the Hilsum-Skandalis bibundle E corresponding to the action map, see Section 5.2. Let $[G \backslash E]_p$ be the subcategory of $[G \backslash E]$ as in Example 4.5, which is a prestack whose stackification is $[G \backslash E]$.

The proof of Prop. 5.5 consists of defining categories fibred in groupoids and morphisms,

$$\mathcal{A}_1 \longrightarrow \mathcal{A}_2 \longrightarrow \mathcal{A}_3 := X_0 \times_{BX/_p BG} X_0,$$

such that

- \mathcal{A}_1 is canonically isomorphic to $[G \backslash E]_p$,
- the morphism $\mathcal{A}_1 \rightarrow \mathcal{A}_2$ is a (strict) isomorphism,
- $\mathcal{A}_2 \rightarrow \mathcal{A}_3$ is a monomorphism and an epimorphism.

By Prop. 2.2 (iv), it follows that the stackification $[G \backslash E] \rightarrow X_0 \times_{BX/_p BG} X_0$ of the induced morphism $[G \backslash E]_p \rightarrow X_0 \times_{BX/_p BG} X_0$ is an isomorphism, and this proves the proposition.

The category \mathcal{A}_1

The objects of \mathcal{A}_1 are triples (U, f, c) , where U is a manifold and $f : U \rightarrow G_0$ and $c : U \rightarrow E$ are morphisms such that $p_t a c = f$, for a in (5.4) and p_t in (5.5). A morphism

$$(\mu, \nu) : (U, f, c) \rightarrow (U_1, f_1, c_1) \tag{5.8}$$

is defined by maps $\mu : U \rightarrow U_1$ and $\nu : U \rightarrow G$ such that $s_G \nu = f_1 \mu$, $t_G \nu = f$ and $\nu \cdot (c_1 \mu) = c$. Here “ \cdot ” denotes the action of G on E (the same notation is used for multiplication in G). The composition of morphisms is given by

$$(\mu_1, \nu_1)(\mu, \nu) = (\mu_1 \mu, \nu \cdot (\nu_1 \mu)).$$

The identity of an object (U, f, c) is given by $(\text{id}_U, 1_G f)$. The category \mathcal{A}_1 is isomorphic (as a fibred category) to $[G \backslash E]_p$ (cf. Example 4.5, replacing X by E and a by $p_t a$).

The category \mathcal{A}_2

An object in \mathcal{A}_2 is of the form $(U, \hat{a}, \hat{b}, f, \varphi)$, where U is a manifold, $\hat{a}, \hat{b} : U \rightarrow X_0$, $f : U \rightarrow G_0$ are morphisms such that $p\hat{a} = tf$, and $\varphi : \hat{a}^*X \cdot f^*G \rightarrow \hat{b}^*X$ is an isomorphism in $BX(U)$. A morphism in \mathcal{A}_2 from $A = (U, \hat{a}, \hat{b}, f, \varphi)$ to $A_1 = (U_1, \hat{a}_1, \hat{b}_1, f_1, \varphi_1)$ is a pair

$$(\mu, \nu) : A \rightarrow A_1 \quad (5.9)$$

where $\mu : U \rightarrow U_1$, $\nu : U \rightarrow G$ and $\hat{a}_1\mu = \hat{a}$, $\hat{b}_1\mu = \hat{b}$, $s_G\nu = f_1\mu$, $t_G\nu = f$, and

$$\begin{array}{ccc} \hat{a}^*X \cdot f^*G & \xrightarrow{\varphi} & \hat{b}^*X \\ \downarrow & & \downarrow \\ \hat{a}_1^*X \cdot f_1^*G & \xrightarrow{\varphi_1} & \hat{b}_1^*X \end{array} \quad (5.10)$$

commutes, where the map $f^*G = U \times_{G_0} G \rightarrow f_1^*G = U_1 \times_{G_0} G$ is given by $(u, g) \mapsto (\mu(u), \nu(u)^{-1}g)$. The maps $\hat{a}^*X \rightarrow \hat{a}_1^*X$ and $\hat{b}^*X \rightarrow \hat{b}_1^*X$ are naturally induced by μ . Composition and identities in the category \mathcal{A}_2 are defined similarly to \mathcal{A}_1 .

The functor $\mathcal{A}_1 \rightarrow \mathcal{A}_2$

At the level of objects, we consider the following assignment $\text{Obj}(\mathcal{A}_1) \rightarrow \text{Obj}(\mathcal{A}_2)$:

$$(U, f, c) \mapsto (U, \hat{a}, \hat{b}, f, \varphi), \quad (5.11)$$

where $\hat{a} = t_pac$ and $\hat{b} = bc$; we now describe how φ is defined.

In order to define $\varphi : \hat{a}^*X \cdot f^*G \rightarrow \hat{b}^*X$, we use the fact that $\hat{a}^*X \cdot f^*G$ can be described as a fibred product as follows. Let

$$Y_0 := X_0 \times_{p,M,t} G_0.$$

Lemma 5.9 *The principal (right) X -bundle $\hat{a}^*X \cdot f^*G$ over U can be identified with the left vertical arrow of the fibred product*

$$\begin{array}{ccc} U \times_{Y_0} E & \longrightarrow & E \\ \downarrow & & \downarrow a \\ U & \xrightarrow{(\hat{a}, f)} & Y_0, \end{array}$$

where $U \times_{Y_0} E$ is equipped with an X -action along the map $U \times_{Y_0} E \rightarrow X_0$, $(u, e) \mapsto b(e)$, given by $(u, e)x = (u, ex)$.

PROOF: The pair of principal bundles

$$(\hat{a}^*X, f^*G) \in BX(U) \times_{M(U)} BG(U)$$

is identified, via the isomorphism $BX(U) \times_{M(U)} BG(U) = B(X \times_M G)(U)$ (see Prop. 2.14), with the principal $(X \times_M G)$ -bundle $\hat{a}^*X \times_U f^*G$. Since $\hat{a}^*X = U \times_{X_0} X$ and $f^*G = U \times_{G_0} G$, one can check the identification $\hat{a}^*X \times_U f^*G = U \times_{Y_0} Y$, where $Y = X \times_M G$. Hence

$$\hat{a}^*X \cdot f^*G = (\hat{a}^*X \times_U f^*G) \otimes_Y E = ((U \times_{Y_0} Y) \times_{Y_0} E)/Y = U \times_{Y_0} E.$$

The reader may verify that the structure maps of the resulting X -bundle are as in the statement. \square

Since \widehat{b}^*X is a trivial principal X -bundle, the isomorphism $\varphi^{-1} : \widehat{b}^*X \rightarrow \widehat{a}^*X \cdot f^*G$ is determined by a section σ of $\widehat{a}^*X \cdot f^*G$ over U such that $bd\sigma = \widehat{b}$, where

$$d : \widehat{a}^*X \cdot f^*G = U \times_{Y_0} E \rightarrow E$$

is the natural projection (note that bd is the moment map of $\widehat{a}^*X \cdot f^*G$). We now define φ by declaring that φ^{-1} is determined by the unique section σ of $\widehat{a}^*X \cdot f^*G$ over U such that

$$d\sigma = c.$$

This last equation shows that c and σ uniquely determine each other, hence (5.11) is a bijection.

Given two objects (U, f, c) and (U_1, f_1, c_1) of \mathcal{A}_1 , we now consider a map from the set of morphisms between these two objects and the set of morphisms between their images, $(U, \widehat{a}, \widehat{b}, f, \varphi)$ and $(U_1, \widehat{a}_1, \widehat{b}_1, f_1, \varphi_1)$, in \mathcal{A}_2 . The map sends a morphism (μ, ν) in \mathcal{A}_1 (as in (5.8)) to (μ, ν) in \mathcal{A}_2 (as in (5.9)). To check that the map is well defined and that it is a bijection, let (μ, ν) be such that $\mu : U \rightarrow U_1$, $\nu : U \rightarrow G$, and $s_G\nu = f_1\mu$, $t_G\nu = f$. We claim that $\nu \cdot (c_1\mu) = c$ if and only if $\widehat{a}_1\mu = \widehat{a}$, $\widehat{b}_1\mu = \widehat{b}$, and the diagram (5.10) commutes. To verify that the condition $\nu \cdot (c_1\mu) = c$ implies that $\widehat{a}_1\mu = \widehat{a}$, first notice that the G -action on E (see (5.6)) is on the fibers of $t_p a$. Then we have

$$\widehat{a}_1\mu = t_p a c_1\mu = t_p a(\nu \cdot c_1\mu) = t_p a c = \widehat{a}.$$

The fact that the condition $\nu \cdot (c_1\mu) = c$ implies that $\widehat{b}_1\mu = \widehat{b}$ follows, similarly, from the fact that the G -action on E is on the fibers of b . It remains to prove that if $\widehat{a}_1\mu = \widehat{a}$ and $\widehat{b}_1\mu = \widehat{b}$ hold, then $\nu \cdot (c_1\mu) = c$ is equivalent to the commutativity of (5.10). We do that by writing the two maps from \widehat{b}^*X to $\widehat{a}_1^*X \cdot f_1^*G$ in diagram (5.10) in terms of the sections σ and σ_1 that induce φ^{-1} and φ_1^{-1} . Using the identifications $\widehat{a}^*X \cdot f^*G = U \times_{Y_0} E$ and $\widehat{a}_1^*X \cdot f_1^*G = U_1 \times_{Y_0} E$, the two maps read:

$$(u, x) \mapsto (\mu(u), c_1(\mu(u))x), \quad (u, x) \mapsto (\mu(u), \nu(u)^{-1}c(u)x),$$

from which the claim follows. Here we use that the maps on the square (5.10) are given as follows: the vertical right map is $(u, x) \mapsto (\mu(u), x)$, the map φ^{-1} is $\varphi^{-1}(u, x) = \sigma(u)x = (u, c(u)x)$, the map φ_1^{-1} is $\varphi_1^{-1}(u_1, x) = \sigma_1(u_1)x = (u_1, c_1(u_1)x)$, and the vertical left map is $(u, e) \mapsto (\mu(u), \nu(u)^{-1}e)$.

The conclusion is that the assignments on objects and morphisms just described define a functor $\mathcal{A}_1 \rightarrow \mathcal{A}_2$ which is a strict isomorphism in $\text{CFG}_{\mathcal{C}}$.

The functor $\Phi : \mathcal{A}_2 \rightarrow \mathcal{A}_3$

Recall that $\mathcal{A}_3 = X_0 \times_{BX/p} BG X_0$. The functor $\Phi : \mathcal{A}_2 \rightarrow \mathcal{A}_3$, at the level of objects, is defined by

$$\Phi(U, \widehat{a}, \widehat{b}, f, \varphi) = (\widehat{a}, [f^*G, \varphi], \widehat{b}).$$

At the level of morphisms, Φ sends a morphism (μ, ν) in \mathcal{A}_2 , from $(U, \widehat{a}, \widehat{b}, f, \varphi)$ to $(U_1, \widehat{a}_1, \widehat{b}_1, f_1, \varphi_1)$, to the pair (μ, μ) ; to verify that (μ, μ) is indeed a morphism between the corresponding images

in \mathcal{A}_3 , we need to show the commutativity of the diagram

$$\begin{array}{ccc} \hat{a}^*X & \xrightarrow{[f^*G, \varphi]} & \hat{b}^*X \\ \downarrow & & \downarrow \\ \hat{a}_1^*X & \xrightarrow{[f_1^*G, \varphi_1]} & \hat{b}_1^*X \end{array}$$

in BX/pBG , where the vertical maps are induced by μ . This follows from the commutativity of (5.10) and the higher coherences $(x1g)$ and $(xg1)$ of the action.

One may check that Φ is a morphism of categories fibred in groupoids, and we now verify that it is a monomorphism and an epimorphism.

Proposition 5.10 *The following holds:*

- (a) *The morphism $\Phi : \mathcal{A}_2 \rightarrow \mathcal{A}_3$ is an epimorphism.*
- (b) *If the action of BG on BX is 1-free then $\Phi : \mathcal{A}_2 \rightarrow \mathcal{A}_3$ is a monomorphism, i.e., its restriction to fibers $\Phi : \mathcal{A}_2(U) \rightarrow \mathcal{A}_3(U)$ over any manifold U is fully faithful.*

PROOF: To show that Φ is an epimorphism, let $(\hat{a}, [Q, \varphi], \hat{b})$ be an object of \mathcal{A}_3 over U . Then Q is a principal right G -bundle over U along some $\rho : Q \rightarrow G_0$. Let $(\iota_\alpha : U_\alpha \rightarrow U)_\alpha$ be an open cover of U trivializing Q . There are trivializations of Q induced by maps $\sigma_\alpha : U_\alpha \rightarrow Q$ such that $\pi_Q \sigma_\alpha = \iota_\alpha$, where $\pi_Q : Q \rightarrow U$ is the projection. Define $f_\alpha := \rho \sigma_\alpha : U_\alpha \rightarrow G_0$. The pullback of $\varphi : \hat{a}^*X \cdot Q \rightarrow \hat{b}^*X$ via ι_α defines a morphism in $BX(U_\alpha)$,

$$\varphi|_{U_\alpha} : (\hat{a}\iota_\alpha)^*X \cdot f_\alpha^*G \rightarrow (\hat{b}\iota_\alpha)^*X,$$

and the restriction of $(\hat{a}, [Q, \varphi], \hat{b})$ to U_α is $(\hat{a}\iota_\alpha, [f_\alpha^*G, \varphi|_{U_\alpha}], \hat{b}\iota_\alpha) = \Phi(U_\alpha, \hat{a}\iota_\alpha, \hat{b}\iota_\alpha, f_\alpha, \varphi|_{U_\alpha})$.

To prove part (b), let $A = (U, \hat{a}, \hat{b}, f, \varphi)$ and $A_1 = (U_1, \hat{a}_1, \hat{b}_1, f_1, \varphi_1)$ be objects of \mathcal{A}_2 over U . The functor Φ induces a map

$$\text{Hom}_{\mathcal{A}_2(U)}(A, A_1) \longrightarrow \text{Hom}_{\mathcal{A}_3(U)}(\Phi(A), \Phi(A_1)),$$

and we must verify that this is a bijection. Note that $\mathcal{A}_3(U)$ is a set, i.e., all its morphisms are the identities. So it is enough to prove the following two assertions:

- (i) If $\Phi(A) \neq \Phi(A_1)$, then there are no morphisms $A \rightarrow A_1$ over U .
- (ii) If $\Phi(A) = \Phi(A_1)$, then there exists a unique morphism $A \rightarrow A_1$ over U .

The claim in (i) is straightforward: if there is a morphism $A \rightarrow A_1$ over U then its image via Φ is a morphism $\Phi(A) \rightarrow \Phi(A_1)$ over U . Hence $\Phi(A) = \Phi(A_1)$, since $\mathcal{A}_3(U)$ is a set.

We now prove (ii). Assume that $\Phi(A) = \Phi(A_1)$. This is equivalent to having $\hat{a}, \hat{b} : U \rightarrow X_0$ and $f, f_1 : U \rightarrow G_0$ such that $tf = p\hat{a} = tf_1$, and morphisms $\varphi : \hat{a}^*X \cdot f^*G \rightarrow \hat{b}^*X$, $\varphi_1 : \hat{a}_1^*X \cdot f_1^*G \rightarrow \hat{b}_1^*X$ in $BX(U)$ such that $[f^*G, \varphi] = [f_1^*G, \varphi_1]$ as morphisms $\hat{a}^*X \rightarrow \hat{b}^*X$ in $BX/pBG(U)$.

Since a morphism $A \rightarrow A_1$ over U is of the form (id_U, ν) , we have to prove that there exists a unique $\nu : U \rightarrow G$ such that $s_G\nu = f_1$, $t_G\nu = f$, and the diagram

$$\begin{array}{ccc} \hat{a}^*X \cdot f^*G & \xrightarrow{\varphi} & \hat{b}^*X \\ \text{id} \cdot \eta \downarrow & & \downarrow \text{id} \\ \hat{a}^*X \cdot f_1^*G & \xrightarrow{\varphi_1} & \hat{b}_1^*X \end{array} \tag{5.12}$$

commutes, where $\eta : f^*G \rightarrow f_1^*G$ is the map described in the definition of \mathcal{A}_2 : $\eta(u, g) = (u, \nu(u)^{-1}g)$. The fact that $[f^*G, \varphi] = [f_1^*G, \varphi_1]$ implies the existence of a morphism $\eta : f^*G \rightarrow f_1^*G$ of principal bundles over U making (5.12) commute, and having such η is equivalent to having ν with the required properties, so that the existence of ν is proved. For uniqueness, given ν, ν_1 with the desired properties it follows that $\text{id} \cdot \eta = \varphi_1^{-1}\varphi = \text{id} \cdot \eta_1$. Hence, since the BG action is 1-free, we deduce that $\eta_1 = \eta$, and we conclude that $\nu_1 = \nu$. \square

5.4 Application: tensor product of stacky bundles

As seen in Section 2.4, if G is a Lie groupoid, P is a manifold carrying a principal right G -action, and Q is manifold carrying a left G -action, we can construct a new manifold $P \otimes_G Q$. This construction is key in the definition of Morita equivalence of Lie groupoids; in particular, it is used in showing that Morita bibundles can be composed. We now extend this discussion to the context of stacky Lie groupoids.

Let us consider the diagram

$$\begin{array}{ccc} & \mathcal{G} & \\ & \Downarrow & \\ \mathcal{X} & \longrightarrow M \longleftarrow & \mathcal{Y}, \end{array} \quad (5.13)$$

where M is a manifold, and \mathcal{G} is a stacky Lie groupoid that acts on the right on a differentiable stack \mathcal{X} and on the left on a differentiable stack \mathcal{Y} . We assume that one of the maps $\mathcal{X} \rightarrow M$ or $\mathcal{Y} \rightarrow M$ is a submersion, so that $\mathcal{X} \times_M \mathcal{Y}$ is also a differentiable stack. Following Example 3.19 (cf. Prop. A.2) there is an induced right action of \mathcal{G} on $\mathcal{X} \times_M \mathcal{Y}$ given (on objects) by

$$(x, y) \cdot g = (xg, g^{-1}y). \quad (5.14)$$

Let us now discuss the property of weak representability of these actions.

Lemma 5.11 *Let E and E' be Hilsum-Skandalis bibundles for the action maps $\mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X}$ and $\mathcal{Y} \times_M \mathcal{G} \rightarrow \mathcal{Y}$, respectively. Then $E \times_{G_0} E'$ is a Hilsum-Skandalis bibundle for the diagonal action $(\mathcal{X} \times_M \mathcal{Y}) \times_M \mathcal{G} \rightarrow \mathcal{X} \times_M \mathcal{Y}$.*

PROOF: Let $\mathcal{X} = BX$, $\mathcal{Y} = BY$, $\mathcal{G} = BG$. A Hilsum-Skandalis bibundle for the projection $BX \times_M BY \times_M BG \rightarrow BX \times_M BG$ is $X \times_M G \times_M Y_0$, and a Hilsum-Skandalis bibundle for the projection $BX \times_M BY \times_M BG \rightarrow BY \times_M BG$ is $X_0 \times_M G \times_M Y$ (see Lemma 2.15). Hence a Hilsum-Skandalis bibundle for the composition with the action map,

$$(\mathcal{X} \times_M \mathcal{Y}) \times_M \mathcal{G} \rightarrow \mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X},$$

is $(X \times_M G \times_M Y_0) \otimes_{X \times_M G} E$; similarly, a Hilsum-Skandalis bibundle for $(\mathcal{X} \times_M \mathcal{Y}) \times_M \mathcal{G} \rightarrow \mathcal{Y} \times_M \mathcal{G} \rightarrow \mathcal{Y}$ is $(X_0 \times_M G \times_M Y) \otimes_{Y \times_M G} E'$. By Lemma 2.16, a Hilsum-Skandalis bibundle for the diagonal action is

$$((X \times_M G \times_M Y_0) \otimes_{X \times_M G} E) \times_{X_0 \times_M G_0 \times_M Y_0} ((X_0 \times_M G \times_M Y) \otimes_{Y \times_M G} E') \cong E \times_{G_0} E',$$

via $[(1, 1, y_0), e], [(x_0, 1, 1), e'] \mapsto (e, e')$. In the last formula it is understood that given elements $[(x, g, y_0), e] \in (X \times_M G \times_M Y_0) \otimes_{X \times_M G} E$ and $[(x_0, g, y), e'] \in (X_0 \times_M G \times_M Y) \otimes_{Y \times_M G} E'$ we can always find representatives of the form $[(1, 1, y_0), e]$ and $[(x_0, 1, 1), e']$, respectively. \square

Lemma 5.12 *If the action-projection map of the action on \mathcal{X} is weakly representable, or the action-projection map of the action on \mathcal{Y} is weakly representable, then so is the action-projection map of the action on $\mathcal{X} \times_M \mathcal{Y}$.*

PROOF: Since the inversion map on \mathcal{G} is an isomorphism, so is the map $\mathcal{Y} \times_{M, \mathbf{t}} \mathcal{G} \longrightarrow \mathcal{G} \times_{\mathbf{s}, M} \mathcal{Y}$, $(y, g) \mapsto (g^{-1}, y)$. So the action-projection map $\Delta_l : \mathcal{G} \times_M \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$, $\Delta_l(g, y) = (gy, y)$, of the left action on \mathcal{Y} is weakly representable if and only if the action-projection map Δ_r , $(y, g) \mapsto (y, g^{-1}y)$, of the induced right action is weakly representable as a consequence of the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{Y} \times_M \mathcal{G} & \longrightarrow & \mathcal{G} \times_M \mathcal{Y} \\ \Delta_r \downarrow & & \downarrow \Delta_l \\ \mathcal{Y} \times \mathcal{Y} & \longrightarrow & \mathcal{Y} \times \mathcal{Y}, \end{array}$$

where the lower map is the isomorphism defined by $(y, y') \mapsto (y', y)$.

Hence it is enough to prove the assertion in the lemma with the weak representability assumption for \mathcal{X} . In this case, by Lemmas 5.7 and 5.8, we have that $[G \backslash E]$ is representable, where E is a Hilsum-Skandalis bibundle representing the action map on \mathcal{X} , and $G \rightrightarrows G_0$ is a groupoid presenting \mathcal{G} . In particular the action of G on E is free and proper. If E' is a Hilsum-Skandalis bibundle of the morphism $\mathcal{Y} \times_M \mathcal{G} \rightarrow \mathcal{Y}$ that sends (y, g) to $g^{-1}y$, then a Hilsum-Skandalis bibundle of the action on $\mathcal{X} \times_M \mathcal{Y}$ is given by $E \times_{G_0} E'$ (see Lemma 5.11). The induced G -action on $E \times_{G_0} E'$ is the diagonal one, and since the action on the factor E is free and proper, the induced action is free and proper as well. Hence $[G \backslash (E \times_{G_0} E')]$ is representable and the proof is complete once we apply Lemmas 5.7 and 5.8 to the diagonal action. \square

We now define the tensor product of stacky bundles.

Proposition 5.13 *Let $\mathcal{G} \rightrightarrows M$ be a stacky Lie groupoid acting on the differentiable stacks \mathcal{X} (on the right) and \mathcal{Y} (on the left). If the \mathcal{G} -action on \mathcal{X} (or \mathcal{Y}) is principal (see Def. 5.1) then the \mathcal{G} -action on $\mathcal{X} \times_M \mathcal{Y}$ makes it into a principal \mathcal{G} -bundle over the quotient*

$$\mathcal{X} \otimes_{\mathcal{G}} \mathcal{Y} := (\mathcal{X} \times_M \mathcal{Y}) / \mathcal{G}.$$

PROOF: It directly follows from Lemma 5.12 and Theorem 5.2. \square

6 Morita equivalence of stacky Lie groupoids

As recalled in Section 2.4, Morita equivalence plays a key role in groupoid theory; e.g. it identifies groupoids that present isomorphic stacks, and a similar role should be played by Morita equivalence of higher groupoids in connection with higher stacks. In this section, we show how our results on principal actions find applications in the theory of Morita equivalence of stacky Lie groupoids, which we develop through stacky bibundles (a parallel theory for 2-groupoids is presented in [28, Sec. 6], relying on combinatorial and simplicial methods).

6.1 Stacky bibundles and Morita equivalence

In order to define Morita equivalence of stacky Lie groupoids, we start with the notion of bibundle.

Definition 6.1 Let $\mathcal{G}_i \rightrightarrows M_i$, for $i = 1, 2$, be cfg-groupoids. A \mathcal{G}_1 - \mathcal{G}_2 -**bibundle** is an object \mathcal{X} in $\text{CFG}_{\mathcal{C}}$ equipped with maps $\mathbf{a}_1 : \mathcal{X} \rightarrow M_1$ and $\mathbf{a}_2 : \mathcal{X} \rightarrow M_2$,

$$\begin{array}{ccc} \mathcal{G}_1 & & \mathcal{G}_2 \\ \Downarrow & & \Downarrow \\ M_1 & \xleftarrow{\mathbf{a}_1} \mathcal{X} \xrightarrow{\mathbf{a}_2} & M_2, \end{array}$$

together with a left action A_1 of \mathcal{G}_1 on the fibers of \mathbf{a}_2 , a right action A_2 of \mathcal{G}_2 on the fibers of \mathbf{a}_1 , and a 2-isomorphism $\tau : A_1(\text{id} \times A_2) \xrightarrow{\sim} A_2(A_1 \times \text{id})$ that makes the two actions commute. Moreover, τ is assumed to satisfy the following higher coherence conditions:

$$\begin{array}{ccc} (g_1 g'_1 x g_2) : & ((g_1 g'_1) x) g_2 \xleftarrow{\tau} (g_1 g'_1) (x g_2) \xleftarrow{\beta_1} g_1 (g'_1 (x g_2)) & \\ & \beta_1 \cdot \text{id} \uparrow & \downarrow \text{id} \cdot \tau \\ & (g_1 (g'_1 x)) g_2 \xleftarrow{\tau} g_1 ((g'_1 x) g_2) & \\ \\ (g_1 x g_2 g'_2) : & ((g_1 x) g_2) g'_2 \xleftarrow{\beta_2} (g_1 x) (g_2 g'_2) \xleftarrow{\tau} g_1 (x (g_2 g'_2)) & \\ & \tau \cdot \text{id} \uparrow & \downarrow \text{id} \cdot \beta_2 \\ & (g_1 (x g_2)) g'_2 \xleftarrow{\tau} g_1 ((x g_2) g'_2) & \\ \\ (1 x g_2) : & 1(x g_2) \xrightarrow{\tau} (1 x) g_2 & (g_1 x 1) : & g_1(x 1) \xrightarrow{\tau} (g_1 x) 1 \\ & \varepsilon_1 \downarrow \swarrow \varepsilon_1 \cdot \text{id} & & \text{id} \cdot \varepsilon_2 \downarrow \swarrow \varepsilon_2 \\ & x g_2 & & g_1 x \end{array}$$

where $g_1, g'_1 \in \mathcal{G}_1$, $x \in \mathcal{X}$, $g_2, g'_2 \in \mathcal{G}_2$ are such that the compositions make sense, 1 is the appropriate groupoid identity, β_1, β_2 are the associativity 2-isomorphisms of the two actions, and $\varepsilon_1, \varepsilon_2$ are the identity 2-isomorphisms of the two actions (see Def. 3.15).

Definition 6.2 A \mathcal{G}_1 - \mathcal{G}_2 -bibundle \mathcal{X} is **biprincipal** if \mathcal{G}_1 and \mathcal{G}_2 are stacky Lie groupoids, \mathcal{X} is a differentiable stack, $\mathcal{X} \rightarrow M_2$ is a principal left \mathcal{G}_1 -bundle, and $\mathcal{X} \rightarrow M_1$ is a principal right \mathcal{G}_2 -bundle.

Remark 6.3 Clearly, one could also consider just right-principal bibundles as generalizations of Hilsum-Skandalis bibundles to the context of stacky Lie groupoids (cf. Section 2.4).

Definition 6.4 Two stacky Lie groupoids \mathcal{G}_1 and \mathcal{G}_2 are **Morita equivalent** if there exists a biprincipal \mathcal{G}_1 - \mathcal{G}_2 -bibundle.

We may also refer to a biprincipal bibundle as a **Morita bibundle**.

Remark 6.5 In groupoid theory, another approach to Morita equivalence, alternative to bibundles, is through “zig-zags” of weak equivalences (see e.g. [34, Sec. 5.4]). The extension of this theory to higher groupoids in various categories (though not encompassing higher Lie groupoids) has been recently presented in [5], and a parallel construction for higher Lie (Banach) groupoids is considered in [42]. The first steps towards checking the equivalence of both approaches have been taken in [28, Sect 6.4], where it is shown that, for Lie 2-groupoids, a Morita bibundle arising from a strict morphism is exactly a weak equivalence, hence Morita equivalence via zig-zag of weak equivalences gives rise to Morita equivalence via bibundles, as in this paper.

Example 6.6 For any stacky Lie groupoid \mathcal{G} , we have that \mathcal{G} itself is a biprincipal \mathcal{G} - \mathcal{G} -bibundle. In particular, \mathcal{G} is Morita equivalent to itself.

One can find interesting examples of Morita equivalence by considering étale stacky Lie groupoids $\mathcal{G} \rightrightarrows M$ associated with non-integrable Lie algebroids, see Example 3.9.

Example 6.7 Let us assume, for simplicity, that \mathcal{G} is an étale stacky Lie groupoid that is transitive (in the sense that it satisfies the property in Corollary 3.12). In this case, Examples 3.31 and 3.34 imply that, for $x \in M$, $s^{-1}(x)$ is a biprincipal \mathcal{G} - \mathcal{G}_x -bibundle (the commutativity of left and right actions follows from the commutativity of left and right multiplications on \mathcal{G}):

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{G}_x \\ \Downarrow & & \Downarrow \\ M & \xleftarrow[t]{s^{-1}(x)} & \{x\}. \end{array} \quad (6.1)$$

Hence \mathcal{G} is Morita equivalent to the stacky Lie group \mathcal{G}_x ,

The previous example extends the well-known property that transitive Lie groupoids are Morita equivalent to their isotropy groups.

The following section presents a concrete Morita equivalence of the type given by (6.1).

6.2 Example from non-integrable Lie algebroids

We consider here stacky Lie groupoids arising from the simplest examples of non-integrable Lie algebroids.

Let $\omega \in \Omega^2(M)$ be a closed 2-form, and consider the Lie algebroid $A_\omega = TM \oplus \mathbb{R}$, with anchor given by the natural projection on TM and bracket on $\Gamma(A_\omega) = \mathcal{X}(M) \oplus C^\infty(M)$ given by

$$[(X, f), (Y, g)] = ([X, Y], \mathcal{L}_X g - \mathcal{L}_Y f + \omega(X, Y)).$$

We assume for simplicity that M is connected and simply-connected.

The integrability of this Lie algebroid is measured by the group of periods

$$\text{Per}(\omega) := \left\{ \int_\sigma \omega, \sigma \in \pi_2(M) \right\} \subset \mathbb{R}.$$

As proven in [15], A_ω is integrable if and only if this group is discrete. Assuming that this is the case and setting $S_\omega := \mathbb{R}/\text{Per}(\omega)$, it is shown in [14] that ω determines a principal S_ω -bundle P_ω over M (when $\text{Per}(\omega) = \mathbb{Z}$, this is the S^1 -bundle known as the *prequantization* of ω); the

associated gauge groupoid $(P_\omega \times P_\omega)/S_\omega$ is the source-simply-connected Lie groupoid integrating A_ω , denoted by G_ω , and we have a Morita bibundle

$$\begin{array}{ccc} G_\omega & & S_\omega \\ \Downarrow & & \Downarrow \\ M & \xleftarrow{P_\omega} & \{*\}. \end{array} \quad (6.2)$$

Following [14], we recall the approach to integrate A_ω via path spaces [15, 45]. We consider the Banach manifold $P(M)$ of paths (of class C^2), and in the product $P(M) \times \mathbb{R}$ we define an equivalence relation $(\gamma, r) \sim (\bar{\gamma}, \bar{r})$ by the condition that there is a homotopy σ (with fixed end points) from γ to $\bar{\gamma}$ such that

$$\bar{r} - r = \int_\sigma \omega.$$

This equivalence relation defines a regular foliation F_ω on $P(M) \times \mathbb{R}$ whose leaf space is a smooth manifold exactly when A_ω is integrable; moreover,

$$G_\omega = (P(M) \times \mathbb{R}) / \sim. \quad (6.3)$$

To complete the diagram (6.2) from this perspective, we fix $x \in M$ and consider the (Banach) submanifolds $P(M)_x \times \mathbb{R}$ and $L(M)_x \times \mathbb{R}$ of $P(M) \times \mathbb{R}$, where $P(M)_x$ is given by paths starting at x and $L(M)_x$ is given by loops based on x . Both submanifolds are saturated by the leaves of the foliation F_ω ; when A_ω is integrable, the corresponding leaf spaces are smooth and

$$P_\omega = (P(M)_x \times \mathbb{R}) / \sim, \quad S_\omega = (L(M)_x \times \mathbb{R}) / \sim, \quad (6.4)$$

and the left and right actions on (6.2) are induced by concatenation of paths.

We will see how this picture extends to the non-integrable case, i.e., we no longer assume that $\text{Per}(\omega) \subset \mathbb{R}$ is discrete (so it can be dense). In this case one may still view the leaf space (6.3) as the differentiable stack

$$\mathcal{G}_\omega := B\text{Hol}_\omega,$$

where $\text{Hol}_\omega \rightrightarrows P(M) \times \mathbb{R}$ is the holonomy groupoid of the foliation F_ω (more precisely, one should restrict the holonomy groupoid to a complete transversal of F_ω , so as to obtain a finite-dimensional Lie groupoid); moreover, it is shown in [49] that \mathcal{G}_ω is an étale stacky Lie groupoid over M .

The fact that M is connected implies that \mathcal{G}_ω is transitive, so we are in the situation of Example 6.7. By fixing $x \in M$, we have a stacky Morita bibundle

$$\begin{array}{ccc} \mathcal{G}_\omega & & \mathcal{S}_\omega \\ \Downarrow & & \Downarrow \\ M & \xleftarrow{\mathcal{P}_\omega} & \{x\}, \end{array} \quad (6.5)$$

where $\mathcal{P}_\omega := \mathfrak{s}^{-1}(x)$ and $\mathcal{S}_\omega := (\mathcal{G}_\omega)_x$ is the isotropy stacky Lie group at x . Note that when A_ω is integrable, all differentiable stacks in the previous diagram are representable, and we recover (6.2). We will now provide an explicit description of the stacky Lie group \mathcal{S}_ω .

Lemma 6.8 *The holonomy groups of the foliation F_ω are trivial, i.e., any two paths in a leaf joining the same points have the same holonomy.*

PROOF: Consider the submersion $q : P(M) \rightarrow M \times M$, $\gamma \mapsto (\gamma(0), \gamma(1))$, and let $\gamma_\alpha \in P(M)$, with endpoints $q(\gamma_\alpha) = (x_\alpha, y_\alpha)$. We may consider a neighborhood V_α of γ_α in $P(M)$ which is diffeomorphic to the product $U_\alpha^0 \times U_\alpha^1 \times W_\alpha$, where U_α^0 and U_α^1 are open balls in M centered at x_α and y_α , and W_α is a convex subset of a Banach space, in such a way that upon this identification q becomes the natural projection $U_\alpha^0 \times U_\alpha^1 \times W_\alpha \rightarrow U_\alpha^0 \times U_\alpha^1$.

Given $\gamma \in V_\alpha$, we can always find a free homotopy $\sigma_\alpha^\gamma(s, t)$ from $\gamma(t)$ to $\gamma_\alpha(t)$ in such a way that the paths spanned by the endpoints, $\sigma_\alpha^\gamma(s, 0)$ and $\sigma_\alpha^\gamma(s, 1)$, follow the straight lines in U_α^0 and U_α^1 linking $\gamma(0)$ to $\gamma_\alpha(0)$ and $\gamma(1)$ to $\gamma_\alpha(1)$, respectively. Note that given another such homotopy $\bar{\sigma}_\alpha^\gamma(s, t)$, with the same boundary conditions, by the Stokes theorem we have that

$$\int_{\sigma_\alpha^\gamma} \omega = \int_{\bar{\sigma}_\alpha^\gamma} \omega.$$

We now consider the submersion

$$\psi_\alpha : V_\alpha \times \mathbb{R} \rightarrow M \times M \times \mathbb{R}, \quad (\gamma, r) \mapsto (\gamma(0), \gamma(1), r + \int_{\sigma_\alpha^\gamma} \omega),$$

where σ_α^γ is a homotopy from γ to γ_α as above.

Claim: Two points in $V_\alpha \times \mathbb{R}$ are in the same fiber of ψ_α if and only if they lie in the same connected component of the intersection of a leaf of F_ω with $V_\alpha \times \mathbb{R}$.

To verify the claim, we first observe that the condition for two points (γ_0, r_0) and (γ_1, r_1) to belong to the same connected component of the intersection of a leaf of F_ω with $V_\alpha \times \mathbb{R}$ is equivalent to the existence of a path $s \mapsto (\gamma_s, r_s)$ in $V_\alpha \times \mathbb{R}$ from (γ_0, r_0) to (γ_1, r_1) such that

$$r_s = r_0 + \int_{\sigma(\gamma_s)} \omega, \tag{6.6}$$

where, for each s , $\sigma(\gamma_s)$ denotes the homotopy (with fixed endpoints) from γ_0 to γ_s defined by $s' \mapsto \gamma_{s'}$, for $0 \leq s' \leq s$. In this case, noticing that

$$\int_{\sigma_\alpha^{\gamma_0}} \omega = \int_{\sigma(\gamma_s)} \omega + \int_{\sigma_\alpha^{\gamma_s}} \omega,$$

we see that $\psi_\alpha(\gamma_s, r_s)$ is constant. On the other hand, $\psi_\alpha(\gamma_0, r_0) = \psi_\alpha(\gamma_1, r_1)$ if and only if γ_0 and γ_1 have the same endpoints (so they are homotopic through a homotopy σ lying in V_α) and

$$r_1 = r_0 + \int_{\sigma_\alpha^{\gamma_0}} \omega - \int_{\sigma_\alpha^{\gamma_1}} \omega = r_0 + \int_\sigma \omega.$$

Defining the path (γ_s, r_s) by $\gamma_s(t) = \sigma(s, t)$ and $r_s = r_0 + \int_{\sigma(\gamma_s)} \omega$, we see that (γ_0, r_0) and (γ_1, r_1) are in the same connected component of the intersection of a leaf of F_ω with $V_\alpha \times \mathbb{R}$. This completes the proof of the claim.

From the previous claim, we conclude that, for a given $r_\alpha \in \mathbb{R}$, we may view ψ_α as the projection on the transverse direction of a foliated chart of F_ω around $(\gamma_\alpha, r_\alpha)$. Taking two such foliated charts, around two nearby points $(\gamma_\alpha, r_\alpha)$ and (γ_β, r_β) on the same leaf, the corresponding holonomy transformation is given by the transverse component of the transition function from chart α to chart β , which we explicitly compute to be

$$(x, y, r) \mapsto (x, y, r + \int_{\sigma_\beta^\gamma} \omega - \int_{\sigma_\alpha^\gamma} \omega) = (x, y, r + \int_\sigma \omega),$$

where σ is any homotopy (with fixed endpoints) from γ_α to γ_β in V_α . If now $s \mapsto \xi(s) = (\gamma_s, r_s)$ is a path in a leaf of F_ω , from (γ_0, r_0) to (γ_1, r_1) , by subdividing ξ and iterating the holonomy transformation of nearby points above we see that its holonomy transformation is

$$(x, y, r) \mapsto (x, y, r + \int_{\sigma(\gamma_1)} \omega),$$

recalling that $\sigma(\gamma_1)$ is the homotopy with fixed endpoints from γ_0 to γ_1 defined by $s \mapsto \gamma_s$. If ξ is a loop, then $r_1 = r_0 + \int_{\sigma(\gamma_1)} \omega$ must equal r_0 , so $\int_{\sigma(\gamma_1)} \omega = 0$. Hence, the holonomy transformation is trivial. \square

It follows from the previous lemma that the holonomy groupoid Hol_ω is given by pairs $(\xi, \bar{\xi}) \in (P(M) \times \mathbb{R}) \times (P(M) \times \mathbb{R})$ such that $\xi \sim \bar{\xi}$ (with the natural pair groupoid structure).

As a differentiable stack, \mathcal{S}_ω is presented by the restriction of Hol_ω to $L(M)_x \times \mathbb{R}$,

$$\{(\xi, \bar{\xi}) \in (L(M)_x \times \mathbb{R}) \times (L(M)_x \times \mathbb{R}) \mid \xi \sim \bar{\xi}\} \rightrightarrows L(M)_x \times \mathbb{R}. \quad (6.7)$$

Since M is simply-connected, we see that the map $\mathbb{R} \rightarrow L(M)_x \times \mathbb{R}$, $r \mapsto (c_x, r)$, where c_x is the constant loop based on x , is a complete transversal to the foliation in $L(M)_x \times \mathbb{R}$, whose leaves are the orbits of (6.7). Upon restriction of (6.7) to this transversal, we obtain a Morita equivalent Lie groupoid presenting \mathcal{S}_ω , given by

$$\{(r, s) \in \mathbb{R}^2 \mid (c_x, r) \sim (c_x, s)\} \rightrightarrows \mathbb{R}.$$

Noticing that $(c_x, r) \sim (c_x, s)$ if and only if there exists $\sigma \in \pi_2(M, x)$ such that $r - s = \int_\sigma \omega$, we see that the previous Lie groupoid agrees with the action groupoid $\text{Per}(\omega) \ltimes \mathbb{R} \rightrightarrows \mathbb{R}$. Hence, as a differentiable stack, \mathcal{S}_ω is the stack quotient $[\mathbb{R}/\text{Per}(\omega)]$. Its stacky Lie group structure is the one induced by the inclusion of abelian groups $\text{Per}(\omega) \rightarrow \mathbb{R}$, as in Example 3.6. We summarize the previous discussion in the following

Proposition 6.9 *The stacky Lie group \mathcal{S}_ω is the strict 2-group $[\mathbb{R}/\text{Per}(\omega)]$ defined by the action groupoid $\text{Per}(\omega) \ltimes \mathbb{R} \rightrightarrows \mathbb{R}$ and the direct product of abelian groups $\text{Per}(\omega) \times \mathbb{R}$.*

It follows (see (6.5)) that the stacky Lie groupoid $\mathcal{G}_\omega = B\text{Hol}_\omega$ arising from the integration construction for the Lie algebroid A_ω is Morita equivalent to the 2-group $[\mathbb{R}/\text{Per}(\omega)]$.

Hence this last proposition extends the smooth situation depicted in (6.2). In particular, \mathcal{P}_ω is a principal $[\mathbb{R}/\text{Per}(\omega)]$ -bundle over M that generalizes the prequantum S^1 -bundle when ω is not integral.

6.3 Properties of Morita equivalence

We now verify two key properties of Morita equivalence of stacky Lie groupoids: (1) It is an equivalence relation; (2) Representability of stacky Lie groupoids is a Morita invariant (in particular, when restricted to Lie groupoids, Def. 6.4 agrees with the usual notion of Morita equivalence in this setting).

Morita equivalence is an equivalence relation

Theorem 6.10 *Morita equivalence between stacky Lie groupoids is an equivalence relation.*

Since Example 6.6 shows that Morita equivalence is a reflexive relation, the result will follow from Lemma 6.11 and Prop. 6.17 below, which show that it is also symmetric and transitive. Some technical work developed in Appendix A.4 will be used to verify transitivity.

Lemma 6.11 *Morita equivalence between stacky Lie groupoids is a symmetric relation.*

PROOF: Let \mathcal{X} be a biprincipal \mathcal{G}_1 - \mathcal{G}_2 -bibundle. We will turn it into a \mathcal{G}_2 - \mathcal{G}_1 -bibundle. By inverting the actions, as in Prop. A.2, we make \mathcal{G}_1 act on \mathcal{X} on the right, and \mathcal{G}_2 act on \mathcal{X} on the left. By Proposition A.7 we see that these new actions are principal as well. We are left to checking that the new actions commute.

Let τ be the 2-isomorphism associated with the commutativity of the two original actions, as in Definition 6.1,

$$\tau(g_1, x, g_2) : g_1(xg_2) \xrightarrow{\sim} (g_1x)g_2,$$

where $x \in \mathcal{X}$ and $g_i \in \mathcal{G}_i$, for $i = 1, 2$, are suitably composable. We define a 2-isomorphism $\bar{\tau}(g_2, x, g_1) : g_2(xg_1) \xrightarrow{\sim} (g_2x)g_1$ by

$$\bar{\tau}(g_2, x, g_1) := \tau(g_1^{-1}, x, g_2^{-1})^{-1}.$$

The higher coherences $(g_1g'_1xg_2)$, $(g_1xg_2g'_2)$, $(1xg_2)$ and (g_1x1) of Definition 6.1 for $\bar{\tau}$ follow, respectively, from conditions $(g_1xg_2g'_2)$, $(g_1g'_1xg_2)$, (g_1x1) , and $(1xg_2)$ for τ . \square

In order to show transitivity of Morita equivalence, we first need two lemmas. The first is a special case of the construction in Prop. 5.13.

Lemma 6.12 *Let a stacky groupoid $\mathcal{G} \rightrightarrows M$ act (on the right) on a stack \mathcal{X} . Consider the induced action of \mathcal{G} on $\mathcal{X} \times_M \mathcal{G}$ by $(x, g) \cdot \bar{g} = (x\bar{g}, \bar{g}^{-1}g)$. Then the original action $\mathbf{A} : \mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X}$, $(x, g) \mapsto x \cdot g$, induces an identification of $(\mathcal{X} \times_M \mathcal{G})/\mathcal{G}$ with \mathcal{X} .*

PROOF: Since the action is principal on the \mathcal{G} -factor, it is also in the product, see Prop. 5.13. The induced action is on the fibers of $\mathbf{A} : \mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X}$ via a natural 2-isomorphism

$$\gamma : xg \rightarrow (x\bar{g})(\bar{g}^{-1}g),$$

given by a composition of structure maps that we leave to the reader to write down. From the universal property of the prequotient (cf. Prop. 4.9), there is an induced map $\Phi : (\mathcal{X} \times_M \mathcal{G})/_{\mathcal{G}} \rightarrow \mathcal{X}$, which reads $(x, g) \mapsto xg$ at the level of objects. At the level of morphisms, Φ sends $[\bar{g}, (b, j)] : (x, g) \rightarrow (x_1, g_1)$ to the composition $(b \cdot j) \circ \gamma$:

$$xg \xrightarrow{\gamma} (x\bar{g})(\bar{g}^{-1}g) \xrightarrow{b \cdot j} x_1g_1.$$

We will prove below that Φ is an epimorphism and a monomorphism, so that, using Prop. 2.2 (iv), we conclude that the stackification $\Phi^\# : (\mathcal{X} \times_M \mathcal{G})/\mathcal{G} \rightarrow \mathcal{X}$ of Φ is an isomorphism, leading to the desired identification.

Since any $x \in \text{Obj}(\mathcal{X})$ is isomorphic to $x \cdot 1 = \Phi(x, 1)$ for an appropriate identity $1 \in \text{Obj}(\mathcal{G})$, we see that Φ is essentially surjective, hence an epimorphism.

We outline the proof of the fact that Φ is a monomorphism. We fix a manifold U , and we work with objects and morphisms over U . We have to prove that the section Φ_U is fully faithful; in other words, given a morphism $b : xg \rightarrow x_1g_1$ in \mathcal{X} , we have to prove that there exists a unique morphism $[\bar{g}, (\bar{b}, j)] : (x, g) \rightarrow (x_1, g_1)$ in $(\mathcal{X} \times_M \mathcal{G})/_{\mathcal{G}}$ such that $b = (\bar{b} \cdot j) \circ \gamma$.

For the existence of $[\bar{g}, (\bar{b}, j)]$ we define $\bar{g} := gg_1^{-1}$, and we take $j : \bar{g}^{-1}g = (gg_1^{-1})^{-1}g \rightarrow g_1$ to be a composition of structure 2-isomorphisms of \mathcal{G} that “eliminate” g . Define $\bar{b} : x \cdot \bar{g} \rightarrow x_1$ to be the composition:

$$x(gg_1^{-1}) \longrightarrow (xg)g_1^{-1} \xrightarrow{b \cdot \text{id}} (x_1g_1)g_1^{-1} \longrightarrow x_1$$

where the unspecified maps are structure 2-isomorphisms. The reader can verify that the naturality of the structure 2-isomorphisms and their higher coherences imply that $b = (\bar{b} \cdot j) \circ \gamma$.

For the uniqueness of $[\bar{g}, (\bar{b}, j)]$, we pick another $[\bar{g}_1, (\bar{b}_1, j_1)] : (x, g) \rightarrow (x_1, g_1)$ such that $b = (\bar{b}_1 \cdot j_1) \circ \gamma_1$. We have to prove that there exists $j_2 : \bar{g} \rightarrow \bar{g}_1$ such that the triangle in $\mathcal{X} \times_M \mathcal{G}$,

$$\begin{array}{ccc} (x, g) \cdot \bar{g} & & \\ \text{id} \cdot j_2 \downarrow & \searrow (\bar{b}, j) & \\ (x, g) \cdot \bar{g}_1 & \xrightarrow{(\bar{b}_1, j_1)} & (x_1, g_1), \end{array}$$

commutes, which is equivalent to asking for the following two triangles, in \mathcal{G} and \mathcal{X} respectively, to commute:

$$\begin{array}{ccc} \bar{g}^{-1}g & & x\bar{g} \\ \text{id}(j_2) \cdot \text{id} \downarrow & \searrow j & \downarrow \text{id} \cdot j_2 \\ \bar{g}_1^{-1}g & \xrightarrow{j_1} & g_1 \end{array} \quad \begin{array}{ccc} & & x\bar{g} \\ & \searrow \bar{b} & \downarrow \text{id} \cdot j_2 \\ & & x\bar{g}_1 \xrightarrow{\bar{b}_1} x_1. \end{array} \quad (6.8)$$

We define an auxiliary $j_0 : \bar{g}^{-1} \rightarrow \bar{g}_1^{-1}$ by the composition

$$\bar{g}^{-1} \longrightarrow (\bar{g}^{-1}g)g^{-1} \xrightarrow{j \cdot \text{id}} g_1g^{-1} \xrightarrow{(j_1 \cdot \text{id})^{-1}} (\bar{g}_1^{-1}g)g^{-1} \longrightarrow \bar{g}_1^{-1},$$

where the unlabeled maps are compositions of structure 2-isomorphisms. Finally, define j_2 as the composition

$$\bar{g} \longrightarrow (\bar{g}^{-1})^{-1} \xrightarrow{i(j_0)} (\bar{g}_1^{-1})^{-1} \longrightarrow \bar{g}_1,$$

that is, j_2 is essentially $i(j_0)$. The reader may check that the two triangles in (6.8) commute: using higher coherences, the commutativity of the left triangle follows from the definition of j_2 , and the commutativity of the right one follows from the hypothesis $(\bar{b} \cdot j) \circ \gamma = (\bar{b}_1 \cdot j_1) \circ \gamma_1$. \square

Remark 6.13 With the assumptions of the last lemma, if we let \mathcal{G} act on $\mathcal{X} \times_M \mathcal{G}$ by multiplying on the right on the \mathcal{G} factor, then this action descends to the quotient $(\mathcal{X} \times \mathcal{G})/\mathcal{G}$ and the identification $\Phi : \mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X}$ of the previous lemma turns out to be \mathcal{G} -equivariant (the action on \mathcal{X} is the original one). The reader may consult Lemmas A.10 and A.9 to see how an action descends to a quotient (observe that we could have started with the $\mathcal{G} \times \mathcal{G}$ action $(x, g) \cdot (\tilde{g}, \bar{g}) = (x\tilde{g}, (\tilde{g}^{-1}g)\bar{g})$, so that Lemma A.10 applies directly). As far as the equivariance of Φ is concerned, we just remark that at the level of objects we have $(\Phi(x, g))\bar{g} = (xg) \cdot \bar{g} \xrightarrow{\sim} x(g\bar{g}) = \Phi(x, g\bar{g}) = \Phi((x, g)\bar{g})$, and further details are left to the reader.

Lemma 6.14 *Let the cfg-groupoid $\mathcal{G}_i \rightrightarrows M_i$ act (on the right) on the category fibred in groupoids \mathcal{X}_i , and let $\mathcal{Z}_i := \mathcal{X}_i/p \mathcal{G}_i$, $i = 1, 2$. Suppose that we are given maps $\mathcal{Z}_i \rightarrow M$, where M is a manifold. Then the product groupoid $\mathcal{G}_1 \times \mathcal{G}_2$ acts naturally on $\mathcal{X}_1 \times_M \mathcal{X}_2$, this action is on the*

fibers of the natural map $\mathcal{X}_1 \times_M \mathcal{X}_2 \rightarrow \mathcal{Z}_1 \times_M \mathcal{Z}_2$, and the induced map $(\mathcal{X}_1 \times_M \mathcal{X}_2)/_p (\mathcal{G}_1 \times \mathcal{G}_2) \rightarrow \mathcal{Z}_1 \times_M \mathcal{Z}_2$ is an isomorphism. In other words, we have an identification

$$(\mathcal{X}_1/_p \mathcal{G}_1) \times_M (\mathcal{X}_2/_p \mathcal{G}_2) \simeq (\mathcal{X}_1 \times_M \mathcal{X}_2)/_p (\mathcal{G}_1 \times \mathcal{G}_2). \quad (6.9)$$

PROOF: Use the identification

$$(\mathcal{X}_1 \times_M \mathcal{X}_2) \times_{(M_1 \times M_2)} (\mathcal{G}_1 \times \mathcal{G}_2) \simeq (\mathcal{X}_1 \times_{M_1} \mathcal{G}_1) \times_M (\mathcal{X}_2 \times_{M_2} \mathcal{G}_2) \quad (6.10)$$

to induce the action map

$$(\mathcal{X}_1 \times_M \mathcal{X}_2) \times_{(M_1 \times M_2)} (\mathcal{G}_1 \times \mathcal{G}_2) \longrightarrow \mathcal{X}_1 \times_M \mathcal{X}_2, \quad (x_1, x_2) \cdot (g_1, g_2) = (x_1 \cdot g_1, x_2 \cdot g_2).$$

Similarly, all the other data for an action are induced “componentwise”, and the higher coherences are checked likewise. Moreover, there is an induced projection $r_1 \times r_2 : \mathcal{X}_1 \times_M \mathcal{X}_2 \rightarrow \mathcal{Z}_1 \times_M \mathcal{Z}_2$, and the fact that the product groupoid acts on the fibers of this map is easily checked componentwise as well.

We are left with checking that the induced map $\Phi : (\mathcal{X}_1 \times_M \mathcal{X}_2)/_p (\mathcal{G}_1 \times \mathcal{G}_2) \rightarrow \mathcal{Z}_1 \times_M \mathcal{Z}_2$ (cf. Prop. 4.9) is an isomorphism. At the level of objects Φ is the identity, so that we have just to check that the fibers of Φ over a manifold U are fully faithful. So take (x_1, x_2) and (\bar{x}_1, \bar{x}_2) , objects in $\mathcal{X}_1 \times_M \mathcal{X}_2$, where here and in the following all the objects are over U , and all the morphisms are over the identity of U . Consider now g_1, g_2, b_1, b_2 , where g_i is an object of \mathcal{G}_i such that the composition $x_i \cdot g_i$ makes sense, and $b_i : x_i \cdot g_i \rightarrow \bar{x}_i$ is a morphism in \mathcal{X}_i . Then $[(g_1, g_2), (b_1, b_2)]$ is a morphism $(x_1, x_2) \rightarrow (\bar{x}_1, \bar{x}_2)$ in $(\mathcal{X}_1 \times_M \mathcal{X}_2)/_p (\mathcal{G}_1 \times \mathcal{G}_2)$, and $[(g_1, b_1), [g_2, b_2]]$ is a morphism $(x_1, x_2) \rightarrow (\bar{x}_1, \bar{x}_2)$ in $\mathcal{Z}_1 \times_M \mathcal{Z}_2$. One can check that the action of Φ on morphisms $(x_1, x_2) \rightarrow (\bar{x}_1, \bar{x}_2)$ is given by

$$[(g_1, g_2), (b_1, b_2)] \mapsto [(g_1, b_1), [g_2, b_2]]$$

and that this is a bijection, as desired. \square

Remark 6.15 With the notation of the previous lemma, assume that the \mathcal{G}_i ’s are stacks, the \mathcal{X}_i ’s are prestacks, and that the actions are 1-free. From Prop. 4.8 it follows that the prequotients $\mathcal{X}_i/_p \mathcal{G}_i$ are prestacks. By stackyfing the identification (6.9), and recalling that the stackification of the product of prestacks is the product of the stackifications (compare with the first lines in the proof of Lemma A.9), we see that the result of Lemma 6.14 is still true, under the new assumptions, if we replace the prequotients by the quotients.

Remark 6.16 The following is a special case of the previous lemma that will be useful. Let $\mathcal{G} \rightrightarrows M$ be a cfg-groupoid, and $\mathcal{X}, \mathcal{Y} \in \text{Obj}(\text{CFG}_{\mathcal{C}})$. Assume that \mathcal{G} acts on the left on the fibers of a morphism $\mathcal{X} \rightarrow N$, where N is a manifold, and that we are given a morphism $\mathcal{Y} \rightarrow N$. Consider the induced left action of \mathcal{G} on the fibers of $\mathcal{X} \times_N \mathcal{Y} \rightarrow \mathcal{Y}$. Then there is a canonical identification

$$(\mathcal{X} \times_N \mathcal{Y})/_p \mathcal{G} \xrightarrow{\sim} (\mathcal{X}/_p \mathcal{G}) \times_N \mathcal{Y},$$

given by the identity on objects, and $[g, (b, c)] \mapsto ([g, b], c)$ on morphisms, where $g \in \mathcal{G}$, $b : xg \rightarrow \bar{x}$ is a morphism in \mathcal{X} , and $c : y \rightarrow \bar{y}$ is a morphism in \mathcal{Y} .

We are now ready to prove the transitivity of Morita equivalence.

Proposition 6.17 *Let $\mathcal{G}_i \rightrightarrows M_i$, $i = 1, 2$, and $\mathcal{G} \rightrightarrows M$ be stacky Lie groupoids, and let \mathcal{X} be a biprincipal \mathcal{G}_1 - \mathcal{G} -bibundle and \mathcal{Y} be a biprincipal \mathcal{G} - \mathcal{G}_2 -bibundle. Then the induced \mathcal{G} -action on $\mathcal{X} \times_M \mathcal{Y}$ (via (5.14)) is principal, and the quotient $\mathcal{Z} := (\mathcal{X} \times_M \mathcal{Y})/\mathcal{G}$ inherits the structure of a biprincipal \mathcal{G}_1 - \mathcal{G}_2 -bibundle.*

PROOF: Let us consider the induced right \mathcal{G} -action on $\mathcal{X} \times_M \mathcal{Y}$, as in (5.14). By Prop. 5.13, the quotient $\mathcal{Z} := (\mathcal{X} \times_M \mathcal{Y})/\mathcal{G}$ is a differentiable stack. We have to show that this stack is a biprincipal \mathcal{G}_1 - \mathcal{G}_2 -bibundle,

$$\begin{array}{ccc} \mathcal{G}_1 & & \mathcal{G}_2 \\ \Downarrow & & \Downarrow \\ M_1 & \longleftarrow \mathcal{Z} \longrightarrow & M_2. \end{array}$$

Similarly to what happens in the context of smooth manifolds, the actions of \mathcal{G}_1 on \mathcal{X} and of \mathcal{G}_2 on \mathcal{Y} lift to (strictly) commuting actions on $\mathcal{X} \times_M \mathcal{Y}$. Moreover, these actions descend to the quotient \mathcal{Z} . The proof of this fact is very similar to the proof of Lemma A.10, and relies on the commutativity of the actions. For instance, the commutativity of the actions of \mathcal{G}_1 and \mathcal{G} over \mathcal{X} implies that the action map $\mathcal{G}_1 \times_{M_1} \mathcal{X} \times_M \mathcal{Y} \rightarrow \mathcal{X} \times_M \mathcal{Y}$ is \mathcal{G} -equivariant, and one applies Remark 4.15(b) to induce the desired action $\mathcal{G}_1 \times_{M_1} \mathcal{Z} \rightarrow \mathcal{Z}$. The coherence 2-isomorphisms and their higher coherence properties are induced and proved as in Lemma A.10. Let us write explicitly how the induced action operates at the level of morphisms (at the levels of objects it acts as the original action: $g_1 \cdot (x, y) = (g_1 \cdot x, y)$). Let $(j_1, [g, (b_1, b_2)]) : (g_1, x, y) \rightarrow (\bar{g}_1, \bar{x}, \bar{y})$ be a morphism in $\mathcal{G}_1 \times_{M_1} \mathcal{Z}$. In particular, $j_1 : g_1 \rightarrow \bar{g}_1$ is a morphism in \mathcal{G}_1 , g is an object of \mathcal{G} , $b_1 : xg \rightarrow \bar{x}$ is a morphism in \mathcal{X} , and $b_2 : g^{-1}y \rightarrow \bar{y}$ is a morphism in \mathcal{Y} . We have $j_1 \cdot [g, (b_1, b_2)] = [g, (j_1 \cdot b_1) \circ \tau_1^{-1}, b_2]$ where $\tau_1 : g_1(xg) \xrightarrow{\sim} (g_1x)g$ is the commutativity 2-isomorphism of the original actions. Finally, it is easy to see that the induced action (strictly) commute.

It remains to show that $\mathcal{Z} \rightarrow M_2$ is a principal \mathcal{G}_1 -bundle and that $\mathcal{Z} \rightarrow M_1$ is a principal \mathcal{G}_2 -bundle. By symmetry, it is enough to prove the assertion for the \mathcal{G}_2 -action.

We know that $\mathcal{Y} \rightarrow M$ is a submersion and an epimorphism, and since these properties are stable under base change (see Prop. 2.11), we conclude that $\mathcal{X} \times_M \mathcal{Y} \rightarrow \mathcal{X}$ is a submersion and an epimorphism. Composing with $\mathcal{X} \rightarrow M_1$, which is a submersion and an epimorphism, we get a map $\mathcal{X} \times_M \mathcal{Y} \rightarrow M_1$ with the same properties. Since the triangle

$$\begin{array}{ccc} \mathcal{X} \times_M \mathcal{Y} & \longrightarrow & \mathcal{Z} \\ & \searrow & \downarrow \\ & & M_1 \end{array}$$

commutes, it follows that $\mathcal{Z} \rightarrow M_1$ is an epimorphism. By Propositions 4.6(a) and 2.11(c), it follows that $\mathcal{Z} \rightarrow M_1$ is a submersion.

Lastly, we must prove that the map $\Delta_{\mathcal{Z}} : \mathcal{Z} \times_{M_2} \mathcal{G}_2 \longrightarrow \mathcal{Z} \times_{M_1} \mathcal{Z}$, $\Delta_{\mathcal{Z}}(z, g_2) = (z, z \cdot g_2)$, is an isomorphism. Let us denote by $r : \mathcal{X} \times_M \mathcal{Y} \rightarrow \mathcal{Z}$ the projection and consider the diagram

$$\begin{array}{ccc}
& \mathcal{Y} \times_M (\mathcal{X} \times_M \mathcal{G}) \times_M \mathcal{Y} & \xrightarrow{F_1} (\mathcal{X} \times_M \mathcal{Y}) \times_{M_1} (\mathcal{X} \times_M \mathcal{Y}) \\
& \downarrow F_2 & \downarrow r \times r \\
\mathcal{X} \times_M \mathcal{Y} \times_{M_2} \mathcal{G}_2 & \xrightarrow{F_3} \mathcal{Y} \times_M \mathcal{X} \times_M \mathcal{Y} & \\
\downarrow r \times \text{id}_{\mathcal{G}_2} & & \\
\mathcal{Z} \times_{M_2} \mathcal{G}_2 & \xrightarrow{\Delta_{\mathcal{Z}}} \mathcal{Z} \times_{M_1} \mathcal{Z}. &
\end{array}$$

Here the fibred product $\mathcal{Y} \times_M (\mathcal{X} \times_M \mathcal{G}) \times_M \mathcal{Y}$ is defined by the map

$$\mathcal{X} \times_M \mathcal{G} \rightarrow M, \quad (x, g) \mapsto s(g)$$

on the right, and

$$\mathcal{X} \times_M \mathcal{G} \rightarrow M, \quad (x, g) \mapsto a(x)$$

on the left, where $a : \mathcal{X} \rightarrow M$ is the moment map for the \mathcal{G} -action on \mathcal{X} . The general strategy of this part of the proof is as follows. We define below the maps F_1, F_2, F_3 , and appropriate actions in such a way that: the diagram 2-commutes, F_1 and F_3 are equivariant isomorphisms, and the vertical maps are quotients. As we will see (using Prop. A.14), the map $\Delta_{\mathcal{Z}}$ is induced from the equivariant map F_1 on quotients, from where it follows that $\Delta_{\mathcal{Z}}$ is an isomorphism (cf. Remark 4.15).

The map F_1 is defined by $F_1(y, x, g, \bar{y}) = (x, y, xg, \bar{y})$, the map F_2 is given by $F_2(y, x, g, \bar{y}) = (g^{-1}y, xg, \bar{y})$, while F_3 is given by $F_3(x, y, g_2) = (y, x, yg_2)$.

As for the actions, we define

- a $(\mathcal{G} \times \mathcal{G})$ -action on $\mathcal{Y} \times_M (\mathcal{X} \times_M \mathcal{G}) \times_M \mathcal{Y}$ by

$$(y, x, g, \bar{y}) \cdot (\tilde{g}, \bar{g}) = (\tilde{g}^{-1}y, x\tilde{g}, (\tilde{g}^{-1}g)\bar{g}, \bar{g}^{-1}\bar{y}),$$

- a $(\mathcal{G} \times \mathcal{G})$ -action on $(\mathcal{X} \times_M \mathcal{Y}) \times_{M_1} (\mathcal{X} \times_M \mathcal{Y})$ by

$$(x, y, \bar{x}, \bar{y}) \cdot (\tilde{g}, \bar{g}) = (x\tilde{g}, \tilde{g}^{-1}y, \bar{x}\bar{g}, \bar{g}^{-1}\bar{y}),$$

- a \mathcal{G} -action on $\mathcal{Y} \times_M \mathcal{X} \times_M \mathcal{Y}$ by

$$(y, x, \bar{y}) \cdot \bar{g} = (\bar{g}^{-1}y, x\bar{g}, \bar{g}^{-1}\bar{y}),$$

- a \mathcal{G} -action on $\mathcal{X} \times_M \mathcal{Y} \times_{M_2} \mathcal{G}_2$ by

$$(x, y, g_2) \cdot \bar{g} = (x\bar{g}, \bar{g}^{-1}y, g_2).$$

The proof now goes as follows:

Step 1: F_1 is $(\mathcal{G} \times \mathcal{G})$ -equivariant and an isomorphism – here we use the hypothesis that the map $\mathcal{X} \times_M \mathcal{G} \rightarrow \mathcal{X} \times_{M_1} \mathcal{X}$, $(x, g) \mapsto (x, xg)$, is an isomorphism.

Step 2: F_3 is \mathcal{G} -equivariant and an isomorphism – here we use the hypothesis that the map $\mathcal{Y} \times_{M_2} \mathcal{G}_2 \rightarrow \mathcal{Y} \times_M \mathcal{Y}$, $(y, g_2) \mapsto (y, yg_2)$, is an isomorphism.

Step 3: $\mathcal{Z} \times_{M_1} \mathcal{Z}$ is the quotient of $(\mathcal{X} \times_M \mathcal{Y}) \times_{M_1} (\mathcal{X} \times_M \mathcal{Y})$ by $\mathcal{G} \times \mathcal{G}$ (by Remark 6.15, noticing that each \mathcal{G} -action is principal, so 1-free).

Step 4: We now observe that F_2 and $r \times \text{id}_{\mathcal{G}_2}$ are quotients, and that $\mathcal{Z} \times_{M_2} \mathcal{G}_2$ is the quotient of $\mathcal{Y} \times_M (\mathcal{X} \times_M \mathcal{G}) \times_M \mathcal{Y}$ by $\mathcal{G} \times \mathcal{G}$. Indeed, we can apply Lemma 6.12 (and Remark 6.16) to conclude that F_2 is the quotient by the \mathcal{G} -action of the first factor of $\mathcal{G} \times \mathcal{G}$ (in the sense of Lemma A.10). Moreover, one can check (see Remark 6.13) that the induced \mathcal{G} -action of the second factor on the quotient $\mathcal{Y} \times_M \mathcal{X} \times_M \mathcal{Y}$ coincides with the one given above. Also, $r \times \text{id}_{\mathcal{G}_2}$ is the quotient by the \mathcal{G} -action by Remark 6.16. Since F_3 is an equivariant isomorphism, and noticing that the action of $\mathcal{G} \times \mathcal{G}$ on $\mathcal{Y} \times_M (\mathcal{X} \times_M \mathcal{G}) \times_M \mathcal{Y}$ is 1-free, we can use Proposition A.14 to conclude that $\mathcal{Z} \times_{M_2} \mathcal{G}_2$ is the desired quotient.

Step 5: The diagram is 2-commutative in the sense that, choosing a quasi-inverse F_4 of F_3 , there is a 2-isomorphism from $(r \times r) \circ F_1$ to $\Delta_{\mathcal{Z}} \circ (r \times \text{id}_{\mathcal{G}_2}) \circ F_4 \circ F_2$. To prove this fact we argue as follows. Choose an object (y, x, g, \bar{y}) in $\mathcal{Y} \times_M \mathcal{X} \times_M \mathcal{G} \times_M \mathcal{Y}$, and call (x_1, y_1, g_2) its image via $F_4 \circ F_2$; we have to define a natural isomorphism $(x, y, xg, \bar{y}) \rightarrow (x_1, y_1, x_1, y_1 g_2)$ between the images of these two objects in $\mathcal{Z} \times_{M_1} \mathcal{Z}$. Since F_4 and F_3 are quasi-inverses, we have a natural isomorphism $(b_1, b_2, b_3) : (g^{-1}y, xg, \bar{y}) \rightarrow (y_1, x_1, y_1 g_2)$ between the images of the above two objects in $\mathcal{Y} \times_M \mathcal{X} \times_M \mathcal{Y}$. Observe that $[g, (b_2, b_1)] : (x, y) \rightarrow (x_1, y_1)$ is an isomorphism in \mathcal{Z} . Moreover, we can consider the image $[1, (b_2, b_3) \circ \varepsilon] : (xg, \bar{y}) \rightarrow (x_1, y_1 g_2)$ in \mathcal{Z} of the isomorphism (b_2, b_3) in $\mathcal{X} \times_M \mathcal{Y}$, see Equation (4.9). We conclude the argument by taking $([g, (b_2, b_1)], [1, (b_2, b_3) \circ \varepsilon])$ as the desired isomorphism in $\mathcal{Z} \times_{M_1} \mathcal{Z}$. \square

Remark 6.18 The previous proposition in fact shows that one can compose right-principal stacky bibundles, extending the usual composition of Hilsum-Skandalis bibundles between Lie groupoids. Just as Lie groupoids, Hilsum-Skandalis bibundles and bibundle morphisms form a (non-strict) 2-category, a natural question is whether these “higher Hilsum-Skandalis bibundles” of stacky Lie groupoids, along with suitable notions of 2-morphisms and 3-morphisms, form a 3-category. Initial steps in verifying this were made in [28, Sections 6, 7] (other ways to build a higher category for higher groupoids are considered in [5, 42]).

Relation with Morita equivalence of Lie groupoids

Theorem 6.10 shows that the notion Morita equivalence of Lie groupoids can be extended to the realm of stacky Lie groupoids. We now observe that this more general equivalence relation does not relate representable stacky Lie groupoids (i.e., ordinary Lie groupoids) to nonrepresentable ones, nor does it provide new Morita equivalences between Lie groupoids. In particular, when restricted to ordinary Lie groupoids, it recovers exactly the usual notion of Morita equivalence.

Lemma 6.19 *Let $\mathcal{G} \rightrightarrows M$ be a stacky Lie groupoid, and let $K \rightrightarrows M$ be a Lie groupoid. Suppose that the diagram*

$$\begin{array}{ccc} K & & \mathcal{G} \\ \Downarrow & & \Downarrow \\ M & \xleftarrow{t} K \xrightarrow{s} & M \end{array}$$

is a biprincipal bibundle (i.e., a Morita equivalence). Then \mathcal{G} is canonically isomorphic to K as a stacky Lie groupoid.

PROOF: Suppose that K is biprincipal K - \mathcal{G} -bibundle as in the diagram. Notice that, since all spaces involved are manifolds except for \mathcal{G} , one only has 2-isomorphisms associated with \mathcal{G} itself. By assumption, the canonical map

$$K \times_{s,M,t} \mathcal{G} \longrightarrow K \times_{t,M,t} K, \quad (k, g) \mapsto (k, kg), \quad (6.11)$$

is an isomorphism. There are left actions of K on $K \times_M \mathcal{G}$ and $K \times_M K$:

$$k \cdot (k', g) = (kk', g), \quad k \cdot (k_1, k_2) = (kk_1, kk_2),$$

and the map (6.11) is equivariant with respect to these actions. One may verify that the projections

$$K \times_M \mathcal{G} \longrightarrow \mathcal{G}, \quad (k, g) \mapsto g,$$

(compare with Remark 6.16) and

$$K \times_M K \longrightarrow K, \quad (k_1, k_2) \mapsto k_1^{-1} k_2,$$

are principal left K -bundles. From Remark 4.15, (6.11) induces an isomorphism

$$F : \mathcal{G} \longrightarrow K,$$

given by $g \mapsto 1 \cdot g$, where $1 = 1_{t(g)}$ is the identity element of K on which g can act.

In order to complete the proof we have to check that F preserves the groupoid structure. The morphism F commutes with the source and the identity by definition of the action, and it commutes with the target because \mathcal{G} acts on the fibers of $t : K \rightarrow M$. To check that F preserves the multiplication, let g_1 and g_2 be objects in \mathcal{G} with $s(g_1) = t(g_2)$. Then

$$F(g_1 g_2) = 1_{t(g_1)}(g_1 g_2) = (1_{t(g_1)} g_1) g_2 = (F(g_1) 1_{t(g_2)}) g_2 = F(g_1)(1_{t(g_2)} g_2) = F(g_1) F(g_2).$$

As for ordinary Lie groupoids, preservation of multiplication and identity yields preservation of the inverse: $F(g^{-1}) = F(g)^{-1}$. Indeed, since there is an isomorphism in \mathcal{G} between gg^{-1} and $1_{t(g)}$ and since K is a manifold, the images $F(gg^{-1})$ and $F(1_{t(g)})$ under F are equal. \square

Proposition 6.20 *Let $\mathcal{G}_i \rightrightarrows M_i$ be stacky Lie groupoids, for $i = 1, 2$, and let \mathcal{X} be a biprincipal \mathcal{G}_1 - \mathcal{G}_2 -bibundle. If \mathcal{G}_1 is a Lie groupoid, then \mathcal{X} is a manifold and \mathcal{G}_2 is a Lie groupoid.*

PROOF: Let $X \rightarrow \mathcal{X}$ be an atlas. Composing with the map $a_2 : \mathcal{X} \rightarrow M_2$ (along which \mathcal{G}_2 acts), we obtain a surjective submersion $X \rightarrow M_2$. Then there is an open cover (U_α) of M_2 such that, for each α , there is a section $U_\alpha \rightarrow X$, and we obtain local sections $U_\alpha \rightarrow \mathcal{X}$ of $\mathcal{X} \rightarrow M_2$. For each α , we have isomorphisms

$$U_\alpha \times_{M_2} \mathcal{X} = U_\alpha \times_{\mathcal{X}} \mathcal{X} \times_{M_2} \mathcal{X} = U_\alpha \times_{\mathcal{X}} (\mathcal{G}_1 \times_{M_1} \mathcal{X}) = U_\alpha \times_{M_1} \mathcal{G}_1,$$

so that each $U_\alpha \times_{M_2} \mathcal{X}$ is a manifold (since \mathcal{G}_1 is a Lie groupoid). The manifolds $V_\alpha := U_\alpha \times_{M_2} \mathcal{X}$ cover \mathcal{X} , in the sense that, for any morphism $Z \rightarrow \mathcal{X}$, where Z is a manifold, the family $(Z_\alpha = V_\alpha \times_{\mathcal{X}} Z)_\alpha$ is an open cover of Z . It follows that \mathcal{X} is a manifold.

Let $\mathcal{X} \otimes_{\mathcal{G}_1} \mathcal{X} = (\mathcal{X} \times_{M_1} \mathcal{X}) / \mathcal{G}_1 \rightrightarrows M_2$ be the gauge groupoid of the principal left \mathcal{G}_1 -bundle $\mathcal{X} \rightarrow M_2$ (see e.g. [34, Sec. 5.1]), which is a Lie groupoid. Then we have biprincipal bibundles

$$\begin{array}{ccccc} \mathcal{X} \otimes_{\mathcal{G}_1} \mathcal{X} & & \mathcal{G}_1 & & \mathcal{G}_2 \\ \Downarrow & & \Downarrow & & \Downarrow \\ M_2 & \longleftarrow \mathcal{X} & \longrightarrow M_1 & \longleftarrow \mathcal{X} & \longrightarrow M_2 \end{array}$$

By Proposition 6.17, we see that $\mathcal{X} \otimes_{\mathcal{G}_1} \mathcal{X}$ inherits the structure of a biprincipal bibundle establishing a Morita equivalence between $\mathcal{X} \otimes_{\mathcal{G}_1} \mathcal{X}$ and \mathcal{G}_2 . By Lemma 6.19 we conclude that \mathcal{G}_2 is a Lie groupoid. \square

A Appendices

A.1 Proofs of properties of submersions

We will need the following lemma, which is just [6, Lem. 2.2]:

Lemma A.1 *Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of stacks, and $Y \rightarrow \mathcal{Y}$ be an epimorphism from a manifold Y . If $Y \times_{\mathcal{Y}} \mathcal{X}$ is representable and the induced morphism of manifolds $Y \times_{\mathcal{Y}} \mathcal{X} \rightarrow Y$ is a submersion, then F is representable.*

Proof of Prop. 2.10. It is clear that either (b), (c) or (d) imply (a). We now check that (a) implies each one of them.

(a) \implies (b): Assume that $\mathcal{X} \rightarrow \mathcal{Y}$ is a submersion and consider atlases $X \rightarrow \mathcal{X}$ and $Y \rightarrow \mathcal{Y}$ such that the induced map of manifolds $Y \times_{\mathcal{Y}} X \rightarrow Y$ is a submersion. By Lemma A.1 it follows that $X \rightarrow \mathcal{Y}$ is representable.

(a) \implies (c): We have to prove that in the cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ V & \longrightarrow & \mathcal{Y} \end{array}$$

the map $Z \rightarrow V$ is a submersion. By assumption, there is a cartesian diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{X}_1 & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathcal{Y} \end{array}$$

in which $X \rightarrow \mathcal{X}$ and $Y \rightarrow \mathcal{Y}$ are atlases, X_1 is a manifold, and $X_1 \rightarrow Y$ is a submersion.

We consider the cartesian diagram

$$\begin{array}{ccccc}
 W_1 & \longrightarrow & W & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & X_1 & \longrightarrow & X & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 \mathcal{X}'_1 & \longrightarrow & \mathcal{X}' & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & \mathcal{X}_1 & \longrightarrow & \mathcal{X} & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 V_1 & \longrightarrow & V & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & Y & \longrightarrow & \mathcal{Y} &
 \end{array}$$

in which W, W_1, V_1 are manifolds. We now verify that $W \rightarrow V$ is a submersion. Indeed, $W_1 \rightarrow V_1$ is a submersion (being the base change of $X_1 \rightarrow Y$), $V_1 \rightarrow V$ is a surjective submersion (being the base change of $Y \rightarrow \mathcal{Y}$), and $W_1 \rightarrow W$ is surjective (being the base change of $V_1 \rightarrow V$).

By considering the fibred product

$$\begin{array}{ccc}
 U' & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 U & \longrightarrow & \mathcal{X}
 \end{array}$$

where U' is a manifold, $U' \rightarrow X$ is a submersion, and $U' \rightarrow U$ is a surjective submersion, we get cartesian diagrams

$$\begin{array}{ccc}
 W' \longrightarrow U' \\
 \downarrow \quad \downarrow \\
 W \longrightarrow X \\
 \downarrow \quad \downarrow \\
 \mathcal{X}' \longrightarrow \mathcal{X} \\
 \downarrow \quad \downarrow \\
 V \longrightarrow \mathcal{Y}
 \end{array}
 \qquad
 \begin{array}{ccc}
 W' \longrightarrow U' \\
 \downarrow \quad \downarrow \\
 Z \longrightarrow U \\
 \downarrow \quad \downarrow \\
 \mathcal{X}' \longrightarrow \mathcal{X} \\
 \downarrow \quad \downarrow \\
 V \longrightarrow \mathcal{Y}
 \end{array}$$

in which W' is a manifold, $W' \rightarrow W$ is a submersion, and $W' \rightarrow Z$ is a surjective submersion. Since $W \rightarrow V$ is a submersion, so is $Z \rightarrow V$, and we are done. (In the last step we are using that the compositions $U' \rightarrow \mathcal{X}$ in the two diagrams are isomorphic, hence the compositions $W' \rightarrow V$ in the two diagrams are equal.)

(a) \implies (d): Since (a) and (b) are equivalent, we may assume that there exists an atlas $X \rightarrow \mathcal{X}$ such that the composition $X \rightarrow \mathcal{Y}$ is representable. If we take the fibred product

$$\begin{array}{ccc}
 U' & \xrightarrow{b} & X \\
 \downarrow c & & \downarrow d \\
 U & \xrightarrow{e} & \mathcal{X}
 \end{array}$$

then U' is a manifold, c is a surjective submersion and b is a submersion. Hence $Fec \simeq Fdb$ is representable. We have to prove that Fe is representable. Let us consider an atlas $Y \rightarrow \mathcal{Y}$ and

the cartesian diagram

$$\begin{array}{ccc} W' & \longrightarrow & U' \\ f \downarrow & & \downarrow c \\ W & \longrightarrow & U \\ g \downarrow & & \downarrow Fe \\ Y & \longrightarrow & \mathcal{Y} \end{array}$$

where W and W' are manifolds, f is surjective and gf is a submersion. It follows that g is a submersion and, by Lemma A.1, Fe is representable.

Proof of Prop. 2.11. To prove (a), let $\mathcal{X} \rightarrow \mathcal{Y}$ and $\mathcal{Y} \rightarrow \mathcal{Z}$ be submersions. By Prop. 2.10 (b), there exists an atlas $X \rightarrow \mathcal{X}$ such that $X \rightarrow \mathcal{Y}$ is representable. By Prop. 2.10 (d), $X \rightarrow \mathcal{Z}$ is representable. Using Prop. 2.10 (b) again, we see that $\mathcal{X} \rightarrow \mathcal{Z}$ is a submersion.

For (b), we have to prove that, if in the cartesian diagram of differentiable stacks

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

the morphism $\mathcal{X} \rightarrow \mathcal{Y}$ is a submersion, then so is $\mathcal{X}' \rightarrow \mathcal{Y}'$. Since $\mathcal{X} \rightarrow \mathcal{Y}$ is a submersion, there exists an atlas $X \rightarrow \mathcal{X}$ such that $X \rightarrow \mathcal{Y}$ is representable. Let $X' = \mathcal{X}' \times_{\mathcal{X}} X$. Then $X' \rightarrow \mathcal{X}'$ is an atlas and $X' \rightarrow \mathcal{Y}'$ is representable. We conclude that $\mathcal{X}' \rightarrow \mathcal{Y}'$ is a submersion.

To prove (c), let $f : X \rightarrow \mathcal{X}$ be an atlas. Since F is a submersion and an epimorphism, Ff is an atlas of \mathcal{Y} . Since $F'F$ is a submersion, then $F'Ff$ is representable, and we conclude that F' is a submersion.

A.2 Inversion and actions

Given a left (resp. right) action of a cfg-groupoid \mathcal{G} on a category fibred in groupoids \mathcal{X} , there is a natural way to use the groupoid inversion to turn it into a right (resp. left) action. This section discusses some technical aspects of this procedure.

Proposition A.2 *Let the cfg-groupoid $\mathcal{G} \rightrightarrows M$ act on the left on a category fibred in groupoids \mathcal{X} . Then the inversion on \mathcal{G} defines a right action of \mathcal{G} on \mathcal{X} by*

$$x \cdot g := g^{-1} \cdot x.$$

The proof follows from a few observations. First, note that we have natural 2-isomorphisms

$$\theta : (hg)^{-1} \rightarrow g^{-1}h^{-1}, \quad \chi : 1^{-1} \rightarrow 1, \quad (\text{A.1})$$

see Lemma A.3 and (A.2) below for definitions. We will need to check the commutativity of a few diagrams involving these 2-isomorphisms, and in order to simplify matters we will follow some ideas from [25].

Let $\mathcal{G} \rightrightarrows M$ be a cfg-groupoid. For all $g \in \mathcal{G}$, multiplication on the right by g defines a functor

$$R_g : \mathfrak{s}^{-1}(\mathfrak{t}(g)) \rightarrow \mathfrak{s}^{-1}(\mathfrak{s}(g)).$$

One can prove, as in [25, Prop. 1.1], that R_g is an equivalence of categories with quasi-inverse $R_{g^{-1}}$. In particular, R_g is fully faithful, and one can deduce the commutativity of a diagram in $\mathbf{s}^{-1}(\mathbf{t}(g))$ by multiplying on the right by g and checking the commutativity of the resulting diagram. Similar statements hold for multiplication on the left.

Lemma A.3 *For all $g, h \in \mathcal{G}$ with $\mathbf{s}(g) = \mathbf{t}(h)$ there exists a unique isomorphism $\theta_{g,h} : (gh)^{-1} \rightarrow h^{-1}g^{-1}$, natural in g, h , for which the following diagram commutes:*

$$\begin{array}{ccc}
((gh)^{-1}g)h & \xrightarrow{(\theta \cdot \text{id})\text{id}} & ((h^{-1}g^{-1})g)h \\
\alpha \uparrow & & \alpha \cdot \text{id} \uparrow \\
(gh)^{-1}(gh) & & (h^{-1}(g^{-1}g))h \\
i_l \downarrow & & i_l \cdot \text{id} \downarrow \\
1 & \xleftarrow{i_l} h^{-1}h \xleftarrow{\rho \cdot \text{id}} & (h^{-1}1)h.
\end{array}$$

The proof of this lemma follows [25] (see comments after Prop. 1.7 in this reference).

Lemma A.4 *For all composable $g, h, l \in \mathcal{G}$ the following diagram commutes:*

$$\begin{array}{ccccc}
(g \cdot hl)^{-1} & \xrightarrow{\theta} & (hl)^{-1}g^{-1} & \xrightarrow{\theta \cdot \text{id}} & l^{-1}h^{-1} \cdot g^{-1} \\
i(\alpha) \downarrow & & & & \uparrow \alpha \\
(gh \cdot l)^{-1} & \xrightarrow{\theta} & l^{-1}(gh)^{-1} & \xrightarrow{\text{id} \cdot \theta} & l^{-1} \cdot h^{-1}g^{-1}
\end{array}$$

where i is the inverse map of the groupoid.

The idea of the proof of this lemma is to first note, as mentioned above, that it is enough to check the commutativity of the diagram obtained after multiplying on the right by $(gh)l$. Moving parentheses and canceling terms of the form $x^{-1}x$ produces a large diagram, the commutativity of which can be checked applying the higher coherences of the groupoids and the commutativity of the diagram in Lemma A.3.

The 2-isomorphism χ in (A.1) is given by the composition

$$\begin{array}{ccc}
1^{-1} & \xleftarrow{\rho} & 1^{-1}1 \\
& \searrow \chi & \downarrow i_l \\
& & 1.
\end{array} \tag{A.2}$$

We will need to use the following compatibilities between θ and χ .

Lemma A.5 *For all $g \in \mathcal{G}$ the following diagrams commute:*

$$\begin{array}{ccc}
(1g)^{-1} & \xrightarrow{\theta} & g^{-1}1^{-1} \\
i(\lambda) \downarrow & & \downarrow \text{id} \cdot \chi \\
g^{-1} & \xleftarrow{\rho} & g^{-1}1
\end{array}
\qquad
\begin{array}{ccc}
(g1)^{-1} & \xrightarrow{\theta} & 1^{-1}g^{-1} \\
i(\rho) \downarrow & & \downarrow \chi \cdot \text{id} \\
g^{-1} & \xleftarrow{\lambda} & 1g^{-1}.
\end{array}$$

PROOF: One can prove the commutativity of both diagrams by multiplying them on the right by g , and using Lemma A.3 and the higher coherences of the groupoid. (For the second diagram, one uses the fact that $\rho = \lambda : 1 \cdot 1 \rightarrow 1$, an equality that can be proven as in [24, Thm. 3].) \square

Remark A.6 In this paper, we have chosen to check the commutativity of all the necessary diagrams directly, by using higher coherence conditions. A more general alternative would be to prove a coherence theorem in the spirit of [25].

To complete the proof of Prop. A.2, take the associativity and identity 2-isomorphisms of the new action as

$$x \cdot gh \xrightarrow{\beta^*} xg \cdot h \quad x1 \xrightarrow{\varepsilon^*} x,$$

where

$$\beta^* : \quad x \cdot gh = (gh)^{-1}x \xrightarrow{\theta \cdot \text{id}} h^{-1}g^{-1} \cdot x \xrightarrow{\beta} h^{-1} \cdot g^{-1}x = xg \cdot h$$

and

$$\varepsilon^* : \quad x1 = 1^{-1}x \xrightarrow{\chi \cdot \text{id}} 1x \xrightarrow{\varepsilon} x.$$

Here β and ε are the 2-isomorphisms of the original action, and θ and χ are as in (A.1).

The higher coherence $(xghl)$ for β^* follows from condition $(lhgx)$ for β , and by the commutativity of the diagram in Lemma A.4. The higher coherence $(x1g)$ for the new action follows from $(g1x)$ for the original action and the commutativity of the first diagram in Lemma A.5. Similarly, the higher coherence $(xg1)$ for the new action follows from $(1gx)$ for the original action and the commutativity of the second diagram in Lemma A.5.

Finally, we remark that, as expected, one can interchange left and right principal bundles by inverting actions.

Proposition A.7 *Let \mathcal{G} be a stacky Lie groupoid and $r : \mathcal{X} \rightarrow \mathcal{S}$ be a morphism of differentiable stacks. Let \mathcal{G} act on the left on \mathcal{X} , and let $x \cdot g := g^{-1} \cdot x$ be the corresponding right action. Then r makes \mathcal{X} into a principal \mathcal{G} -bundle for the left action if and only if it does for the right action.*

PROOF: We must check conditions 1., 2. and 3. of Definition 3.24. Condition 1. is immediate. To analyze 2., let γ be the 2-isomorphism

$$\gamma(g, x) : r(x) \xrightarrow{\sim} r(gx),$$

and let γ' be the corresponding 2-isomorphism for the right action:

$$\gamma'(x, g) := \gamma(g^{-1}, x) : r(x) \xrightarrow{\sim} r(xg).$$

We have to show that γ makes \mathcal{G} act on the fibers of r via the left action if and only if γ' makes \mathcal{G} act on the fibers of r via the right action. We refer to $(\gamma\beta)$ and $(\gamma\varepsilon)$ as the higher coherences for the left action, and $(\gamma\beta)'$ and $(\gamma\varepsilon)'$ the analogous conditions for the right action. Consider the diagram

$$\begin{array}{ccccc} r(x) & \xrightarrow{r(\text{id})} & r(x) & \xrightarrow{\gamma} & r(g^{-1}x) \\ \gamma \downarrow & & \downarrow \gamma & & \downarrow \gamma \\ r((gh)^{-1}x) & \xrightarrow{r(\theta \cdot \text{id})} & r((h^{-1}g^{-1})x) & \xrightarrow{r(\beta)} & r(h^{-1}(g^{-1}x)) \end{array}$$

where $\theta : (gh)^{-1} \rightarrow h^{-1}g^{-1}$ is the isomorphism in (A.1). The outer rectangle is the one of condition $(\gamma\beta)'$, while the right square is the one of $(\gamma\beta)$. Moreover, the left square is commutative by naturality of γ . Hence, condition $(\gamma\beta)$ implies $(\gamma\beta)'$. For the converse, one has to substitute g and h in the above diagram with g^{-1} and h^{-1} . In order to prove that $(\gamma\beta)'$ implies $(\gamma\beta)$, one has to use that there is a 2-isomorphism $i \circ i \rightarrow \text{id}_{\mathcal{G}}$, where i is the inverse of the groupoid, as well as the naturality of β and γ . With a similar argument the reader can check the equivalence between $(\gamma\varepsilon)$ and $(\gamma\varepsilon)'$.

For condition 3., we notice the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{X} \times_M \mathcal{G} & \longrightarrow & \mathcal{X} \times_S \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{G} \times_M \mathcal{X} & \longrightarrow & \mathcal{X} \times_S \mathcal{X} \end{array}$$

where the left vertical map is the isomorphism $(x, g) \mapsto (g^{-1}, x)$, the right vertical one is the isomorphism $(x, \sigma, y) \mapsto (y, \sigma^{-1}, x)$, the upper horizontal one is $(x, g) \mapsto (x, \gamma', xg)$, and the lower horizontal one is $(g, x) \mapsto (gx, \gamma^{-1}, x)$. \square

A.3 The prequotient construction

Here we present the **proof of Prop. 4.1**.

We first show that $\mathcal{X}/_p \mathcal{G}$ is a category, which involves proving that the associativity and the identity axioms for composition of morphisms hold. In order to illustrate the general proof method, we give a detailed account of the associativity axiom for morphisms in $\mathcal{X}/_p \mathcal{G}$, leaving the identity axioms to the reader. We remark that this is a point where the higher coherence conditions (a4) for the action of \mathcal{G} on \mathcal{X} must be used. Indeed, the associativity for composition of morphisms relies on $(xghl)$, while $(xg1)$ and $(x1g)$ are used to prove the left and right identity axioms, respectively.

Associativity of the composition of morphisms in $\mathcal{X}/_p \mathcal{G}$: Consider three morphisms in $\mathcal{X}/_p \mathcal{G}$,

$$x \xrightarrow{[g,b]} y \xrightarrow{[h,c]} z \xrightarrow{[k,d]} w.$$

In order to perform the various compositions we arbitrarily choose pull backs in \mathcal{G} :

$$\begin{array}{ccccc} \mu^* h & \xrightarrow{\mu_h} & h & & \nu^* k & \xrightarrow{\nu_k} & k & & (\nu\mu)^* k & \xrightarrow{(\nu\mu)_k} & k \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow{\mu} & V & & V & \xrightarrow{\nu} & W & & U & \xrightarrow{\nu\mu} & W \end{array}$$

where $\mu := \pi_{\mathcal{X}}(b)$ and $\nu := \pi_{\mathcal{X}}(c)$. Then we have

$$\begin{aligned} [h, c][g, b] &= [g \cdot \mu^* h, c \circ (b \cdot \mu_h) \circ \beta(x, g, \mu^* h)], \\ [k, d][h, c] &= [h \cdot \nu^* k, d \circ (c \cdot \nu_k) \circ \beta(y, h, \nu^* k)]. \end{aligned}$$

In order to perform compositions using three morphisms, we have to choose pullbacks in \mathcal{G} :

$$\begin{array}{ccc} \mu^*(h \cdot \nu^* k) & \xrightarrow{\mu_h \cdot \nu^* k} & h \cdot \nu^* k \\ \downarrow & & \downarrow \\ U & \xrightarrow{\mu} & V \end{array} \quad \begin{array}{ccc} \mu^* \nu^* k & \xrightarrow{\mu \nu^* k} & \nu^* k \\ \downarrow & & \downarrow \\ U & \xrightarrow{\mu} & V, \end{array}$$

but this time we do not make arbitrary choices; instead, we set $\mu^*(h \cdot \nu^*k) := \mu^*h \cdot (\nu\mu)^*k$, and $\mu_h \cdot \nu^*k := \mu_h \cdot \mu_{\nu^*k}$, and $\mu^*\nu^*k := (\nu\mu)^*k$, and μ_{ν^*k} to be the unique map over μ such that $\nu_k \mu_{\nu^*k} = (\nu\mu)_k$. Then the compositions read as follows:

$$([k, d][h, c])[g, b] = [g \cdot (\mu^*h \cdot (\nu\mu)^*k), d \circ (c \cdot \nu_k) \circ \beta(y, h, \nu^*k) \circ (b \cdot (\mu_h \cdot \mu_{\nu^*k})) \circ \beta(x, g, \mu^*h \cdot (\nu\mu)^*k)]$$

$$[k, d]([h, c][g, b]) = \left[(g \cdot \mu^*h) \cdot (\nu\mu)^*k, d \circ \left((c \circ (b \cdot \mu_h) \circ \beta(x, g, \mu^*h)) \cdot (\nu\mu)_k \right) \circ \beta(x, g \cdot \mu^*h, (\nu\mu)^*k) \right].$$

In order to prove the equality between the two ways of composing the three arrows, we consider the associativity isomorphism in \mathcal{G} defined in **(g3)**:

$$\alpha := \alpha(g, \mu^*h, (\nu\mu)^*k) : g \cdot (\mu^*h \cdot (\nu\mu)^*k) \xrightarrow{\sim} (g \cdot \mu^*h) \cdot (\nu\mu)^*k.$$

Since $\pi_{\mathcal{G}}(\alpha) = \text{id}_U$, it follows that $\mathbf{t}(\alpha) = \text{id}_{\mathbf{a}(x)}$.

Note that, by the functoriality of the multiplication \mathbf{m} on \mathcal{G} , the following holds:

$$(\bar{j}_2 \circ \bar{j}_1) \cdot (j_2 \circ j_1) = (\bar{j}_2 \cdot j_2) \circ (\bar{j}_1 \cdot j_1),$$

where $j_1, j_2, \bar{j}_1, \bar{j}_2$ are morphisms

$$g_1 \xrightarrow{j_1} g_2 \xrightarrow{j_2} g_3, \quad \bar{g}_1 \xrightarrow{\bar{j}_1} \bar{g}_2 \xrightarrow{\bar{j}_2} \bar{g}_3$$

for which the compositions and multiplications above make sense. Considering the expression for $([k, d][h, c])[g, b]$, we then have

$$(c \circ (b \cdot \mu_h) \circ \beta(x, g, \mu^*h)) \cdot (\nu\mu)_k = (c \cdot \nu_k) \circ ((b \cdot \mu_h) \cdot \mu_{\nu^*k}) \circ (\beta(x, g, \mu^*h) \cdot \text{id}_{(\nu\mu)^*k}).$$

Then proving associativity boils down to proving that α makes the following diagram (which is (4.3) in our context) commute:

$$\begin{array}{ccccc} x \cdot (g \cdot (\mu^*h \cdot (\nu\mu)^*k)) & \xrightarrow{\beta} & (x \cdot g) \cdot (\mu^*h \cdot (\nu\mu)^*k) & \xrightarrow{b \cdot (\mu_h \cdot \mu_{\nu^*k})} & y \cdot (h \cdot \nu^*k) \\ \text{id} \cdot \alpha \downarrow & & \downarrow \beta & & \downarrow \beta \\ x \cdot ((g \cdot \mu^*h) \cdot (\nu\mu)^*k) & & & & \\ \downarrow \beta & & & & \downarrow \beta \\ (x \cdot (g \cdot \mu^*h)) \cdot (\nu\mu)^*k & \xrightarrow{\beta \cdot \text{id}} & ((x \cdot g) \cdot \mu^*h) \cdot (\nu\mu)^*k & \xrightarrow{(b \cdot \mu_h) \cdot \mu_{\nu^*k}} & (y \cdot h) \cdot \nu^*k \xrightarrow{c \cdot \nu_k} z \cdot k \\ & & & & \uparrow d \\ & & & & w \end{array}$$

The commutativity of the left square is due to the higher coherence $(xghl)$, whereas the commutativity of the right square follows from the naturality of β . This finishes the proof of the associativity of the composition of morphisms $\mathcal{X}/_p \mathcal{G}$. As already mentioned, the identity axioms are proven similarly, and we conclude that $\mathcal{X}/_p \mathcal{G}$ is a category.

The fact that $\pi_{\mathcal{X}/_p \mathcal{G}}$ in (4.7) is a functor follows from definitions (4.4), (4.5) and (4.6), and from the fact that, by the definition of natural transformation of fibred categories in groupoids, $\pi_{\mathcal{X}}(\beta)$ and $\pi_{\mathcal{X}}(\varepsilon)$ are sent to identities in \mathcal{C} .

Finally, we must prove that $\mathcal{X}/_p \mathcal{G}$ is fibred in groupoids over \mathcal{C} . We outline the proof, leaving the details to the reader. Given $\mu : U \rightarrow V$ in \mathcal{C} and $y \in \mathcal{X}$ with $\pi_{\mathcal{X}}(y) = V$, we first have to

produce a cartesian arrow $x \rightarrow y$ in $\mathcal{X}/_p \mathcal{G}$ over μ ; this proves that $\mathcal{X}/_p \mathcal{G}$ is fibred over \mathcal{C} . To do that we take an arrow $b : x \rightarrow y$ in \mathcal{X} over μ and define an arrow in $\mathcal{X}/_p \mathcal{G}$ over μ by

$$[1_{\mathbf{a}(x)}, b \circ \varepsilon(x)] : x \longrightarrow y.$$

A direct computation shows that this arrow is cartesian; we remark that the higher coherence $(xg1)$ in **(a4)** has to be used.

In order to prove that $\mathcal{X}/_p \mathcal{G}$ is fibred in groupoids, we have to check that any morphism $[g, b] : x \rightarrow y$ in $\mathcal{X}/_p \mathcal{G}$ over an identity of \mathcal{C} is an isomorphism. We claim that the inverse is given by

$$[g, b]^{-1} = [g^{-1}, \varepsilon(x) \circ (\text{id}_x \cdot 1_r(g)) \circ \beta(x, g, g^{-1})^{-1} \circ (b^{-1} \cdot \text{id}_{g^{-1}})].$$

The proof of the claim is direct; we remark that in proving that $[g, b][g, b]^{-1} = \text{id}_y$ we use the higher coherences $(xghl)$, $(x1g)$, $(xg1)$ of **(a4)**, and $(gg^{-1}g)$ of **(g4)**.

This concludes the proof of Prop. 4.1.

We will need the next observation in Prop. 4.8.

Remark A.8 Given an arrow $[g, b]$ in $\mathcal{X}/_p \mathcal{G}$ over an object V of \mathcal{C} and an arrow $\mu : U \rightarrow V$ in \mathcal{C} then $\mu^*[g, b] = [\mu^*g, \mu^*b]$.

A.4 Quotient in stages

In this section we consider actions by products of cfg-groupoids. The main result, used in Prop. 6.17, asserts that in this case quotients can be taken in stages, i.e., with respect to one factor at a time.

Lemma A.9 *Let \mathcal{Y} be a prestack that carries a (right) action of a stacky groupoid \mathcal{G} along $\mathbf{a} : \mathcal{Y} \rightarrow M$. There is an induced \mathcal{G} -action on the stackification \mathcal{Y}^\sharp in such a way that the stackification map $\mathbf{b} : \mathcal{Y} \rightarrow \mathcal{Y}^\sharp$ is \mathcal{G} -equivariant. Moreover, if the \mathcal{G} -action on \mathcal{Y} is 1-free then so is the action on \mathcal{Y}^\sharp .*

PROOF: First of all we observe that $\mathbf{b} \times \text{id} : \mathcal{Y} \times_M \mathcal{G} \rightarrow \mathcal{Y}^\sharp \times_M \mathcal{G}$ is a stackification map. Indeed, $\mathcal{Y} \times_M \mathcal{G}$ is a prestack, $\mathcal{Y}^\sharp \times_M \mathcal{G}$ is a stack, and one can check that $\mathbf{b} \times \text{id}$ is a monomorphism and an epimorphism, hence the observation follows from Proposition 2.2 (iv). (Here we use that \mathcal{Y} is a prestack and not merely a cfg.)

One obtains all the data and higher coherences for the \mathcal{G} -action on \mathcal{Y}^\sharp from the universal property of the stackification. We sketch the main steps leaving the details to the reader. There is an induced moment map $\mathbf{a}^\sharp : \mathcal{Y}^\sharp \rightarrow M$ such that $\mathbf{a}^\sharp \mathbf{b} = \mathbf{a}$. Applying the universal property of the stackification map $\mathbf{b} \times \text{id}$ (part (i) of Prop. 2.2) to the solid diagram:

$$\begin{array}{ccc} \mathcal{Y} \times_M \mathcal{G} & \xrightarrow{\mathbf{A}} & \mathcal{Y} \\ \mathbf{b} \times \text{id} \downarrow & & \downarrow \mathbf{b} \\ \mathcal{Y}^\sharp \times_M \mathcal{G} & \xrightarrow[\mathbf{A}^\sharp]{} & \mathcal{Y}^\sharp, \end{array}$$

where \mathbf{A} is the original action, one gets a pair $(\mathbf{A}^\sharp, \delta)$, where $\delta : \mathbf{A}^\sharp(\mathbf{b} \times \text{id}) \rightarrow \mathbf{b}\mathbf{A}$. The 2-isomorphism δ is the one that makes \mathbf{b} equivariant. The associativity 2-isomorphism $\beta^\sharp : \mathbf{A}^\sharp(\text{id} \times \mathbf{m}) \rightarrow \mathbf{A}^\sharp(\mathbf{A}^\sharp \times \text{id})$ of the new action is induced by the $\beta : \mathbf{A}(\text{id} \times \mathbf{m}) \rightarrow \mathbf{A}(\mathbf{A} \times \text{id})$ of the original

action by applying the universal property of the stackification map $\mathbf{b} \times \text{id} \times \text{id}$ (part (ii) of Prop. 2.2) to the diagram:

$$\begin{array}{ccc} \mathcal{Y} \times_M \mathcal{G} \times_M \mathcal{G} & \xrightarrow[\mathbf{A}(\mathbf{A} \times \text{id})]{\mathbf{A}(\text{id} \times \mathbf{m})} & \mathcal{Y} \\ \mathbf{b} \times \text{id} \times \text{id} \downarrow & & \downarrow \mathbf{b} \\ \mathcal{Y}^\# \times_M \mathcal{G} \times_M \mathcal{G} & \xrightarrow[\mathbf{A}^\#(\mathbf{A}^\# \times \text{id})]{\mathbf{A}^\#(\text{id} \times \mathbf{m})} & \mathcal{Y}^\#. \end{array}$$

The uniqueness part of (ii) of Prop. 2.2 implies the equivariance higher coherence $(\delta\beta_1\beta_2)$, see Def. 3.20. The higher coherence $(xghl)$ for the new action (the “pentagon”) follows from the same coherence for the old action, the already proven $(\delta\beta_1\beta_2)$, and the universal property of the stackification map $\mathbf{b} \times \text{id} \times \text{id} \times \text{id}$ (the uniqueness part of Prop. 2.2, part (ii)). The same ideas can be used to construct the identity 2-isomorphism $\varepsilon^\#$ for the new action, and to prove first the equivariance higher coherence $(\delta\varepsilon_1\varepsilon_2)$, and then the action higher coherences $(x1g)$ and $(xg1)$.

For the last assertion of the lemma, recall that the stackification map $\mathbf{b} : \mathcal{Y} \rightarrow \mathcal{Y}^\#$ the stackification map is a monomorphism and an epimorphism (see Prop. 2.2 (iii)), and that it is \mathcal{G} -equivariant. Take $y \in \mathcal{Y}^\#$ and $j, j' : g \rightarrow \bar{g}$ in \mathcal{G} , where the objects are over the same manifold U , and the morphisms are over id_U . Assuming that $\text{id}_y \cdot j = \text{id}_y \cdot j'$, we have to prove that $j = j'$. There exist an open cover (U_α) of U , objects $y_\alpha \in \mathcal{Y}_{U_\alpha}$, and isomorphisms $\mathbf{b}(y_\alpha) \rightarrow y|_{U_\alpha}$. We have $\mathbf{b}(\text{id}_{y_\alpha} \cdot j|_{U_\alpha}) = \text{id}_{\mathbf{b}(y_\alpha)} \cdot j|_{U_\alpha} = \text{id}_{\mathbf{b}(y_\alpha)} \cdot j'|_{U_\alpha} = \mathbf{b}(\text{id}_{y_\alpha} \cdot j'|_{U_\alpha})$, where the second equality follows from $\text{id}_{y|_{U_\alpha}} \cdot j|_{U_\alpha} = \text{id}_{y|_{U_\alpha}} \cdot j'|_{U_\alpha}$ using the isomorphisms $\mathbf{b}(y_\alpha) \rightarrow y|_{U_\alpha}$. Hence, $\text{id}_{y_\alpha} \cdot j|_{U_\alpha} = \text{id}_{y_\alpha} \cdot j'|_{U_\alpha}$ (\mathbf{b} being a monomorphism), and since the \mathcal{G} -action on \mathcal{Y} is 1-free, it follows that $j|_{U_\alpha} = j'|_{U_\alpha}$. Since \mathcal{G} is a stack, we conclude that $j = j'$, as needed. \square

Let \mathcal{G}_i be a cfg-groupoid over M_i , $i = 1, 2$. Suppose that \mathcal{X} is a category fibred in groupoids carrying a (right) action of the product $\mathcal{G} := \mathcal{G}_1 \times \mathcal{G}_2$ along the map $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2) : \mathcal{X} \rightarrow M_1 \times M_2$.

Lemma A.10 *The \mathcal{G} -action on \mathcal{X} restricts to an action of \mathcal{G}_1 on \mathcal{X} (on the fibers of \mathbf{a}_2), and there is an induced action of \mathcal{G}_2 on $\mathcal{X}/_p \mathcal{G}_1$. (Clearly, the same holds for \mathcal{G}_1 and \mathcal{G}_2 interchanged.)*

PROOF: The action of \mathcal{G}_1 on \mathcal{X} is defined in terms of the action of $\mathcal{G}_1 \times \mathcal{G}_2$ by

$$x \cdot g_1 = x \cdot (g_1, 1_{\mathbf{a}_2(x)}),$$

and this restriction process applies to all the data and higher coherences of an action; moreover, since $\mathbf{a}_2(x \cdot g_1) = \mathbf{s}(1_{\mathbf{a}_2(x)}) = \mathbf{a}_2(x)$, the action is on the fibers of $\mathbf{a}_2 : \mathcal{X} \rightarrow M_2$.

Define $\mathcal{Y} := \mathcal{X}/_p \mathcal{G}_1$, and let $\mathbf{q}_1 : \mathcal{X} \rightarrow \mathcal{Y}$ be the quotient map. By the universal property of the prequotient (Prop. 4.9) we have an induced map $\mathcal{Y} \rightarrow M_2$. Consider the map $\mathbf{q}_1 \times \text{id}_{\mathcal{G}_2} : \mathcal{X} \times_{M_2} \mathcal{G}_2 \rightarrow \mathcal{Y} \times_{M_2} \mathcal{G}_2$. In the sequel we use Remark 6.16 to obtain an action of \mathcal{G}_1 on $\mathcal{X} \times_{M_2} \mathcal{G}_2$, and to identify $(\mathcal{X} \times_{M_2} \mathcal{G}_2)/_p \mathcal{G}_1$ with $\mathcal{Y} \times_{M_2} \mathcal{G}_2$. The action map $\mathcal{X} \times_{M_2} \mathcal{G}_2 \rightarrow \mathcal{X}$ is \mathcal{G}_1 -equivariant; indeed, the equivariance 2-isomorphism δ is given by the composition

$$\begin{array}{ccc} (xg_2)g_1 = [x \cdot (1, g_2)] \cdot (g_1, 1) & \xleftarrow{\beta} x \cdot [(1, g_2) \cdot (g_1, 1)] = x \cdot (1g_1, g_21) & \xrightarrow{\text{id}(\lambda_1, \rho_2)} x \cdot (g_1, g_2) \\ \delta \downarrow & & \downarrow \text{id} \\ (xg_1)g_2 = [x \cdot (g_1, 1)] \cdot (1, g_2) & \xleftarrow{\beta} x \cdot [(g_1, 1) \cdot (1, g_2)] = x \cdot (g_11, 1g_2) & \xrightarrow{\text{id}(\rho_1, \lambda_2)} x \cdot (g_1, g_2), \end{array} \quad (\text{A.3})$$

and we leave the verification of the equivariance higher coherences to the reader. It follows from Prop. 4.13, part (i), that there is an induced morphism $A_Y : \mathcal{Y} \times_{M_2} \mathcal{G}_2 \rightarrow \mathcal{Y}$, together with a 2-isomorphism

$$\varphi : \mathbf{q}(x) \cdot g_2 \xrightarrow{\sim} \mathbf{q}(x \cdot g_2), \quad (x, g_2) \in \mathcal{X} \times_{M_2} \mathcal{G}_2.$$

The uniqueness property of the pair (A_Y, φ) (see Prop. 4.13, part (ii)) induces the extra data making A_Y into an action. We will illustrate how this works for the associativity 2-isomorphism. Note that the techniques involved are very similar to the ones in Lemma A.9, due to the formal similarity between the universal property of the stackification (Prop. 2.2) and the universal properties of the quotients (Prop. 4.9) and of equivariant maps (Prop. 4.13). Before moving on, we recall that $\text{Obj}(\mathcal{X}) = \text{Obj}(\mathcal{X}/_p \mathcal{G}_1)$ and observe that the action of \mathcal{G}_2 on $\mathcal{X}/_p \mathcal{G}_1$ agrees with the action on \mathcal{X} on objects.

Consider the diagram

$$\begin{array}{ccc} \mathcal{X} \times_{M_2} \mathcal{G}_2 \times_{M_2} \mathcal{G}_2 & \xrightarrow{F_1, F_2} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y} \times_{M_2} \mathcal{G}_2 \times_{M_2} \mathcal{G}_2 & \xrightarrow{\Phi_1, \Phi_2} & \mathcal{Y} \end{array}$$

where $F_1(x, g, \bar{g}) = x(g\bar{g})$, $F_2(x, g, \bar{g}) = (xg)\bar{g}$, $\Phi_1(y, g, \bar{g}) = y(g\bar{g})$, $\Phi_2(y, g, \bar{g}) = (yg)\bar{g}$. We have 2-isomorphisms (induced by φ)

$$\varphi_1 : \mathbf{q}(x) \cdot (g\bar{g}) \xrightarrow{\sim} \mathbf{q}(x \cdot (g\bar{g})), \quad \varphi_2 : (\mathbf{q}(x)g) \cdot \bar{g} \xrightarrow{\sim} \mathbf{q}((xg) \cdot \bar{g}).$$

Similarly to the equivariance of the action map $\mathcal{X} \times_{M_2} \mathcal{G}_2 \rightarrow \mathcal{X}$, one verifies that F_1 and F_2 are \mathcal{G}_1 -equivariant, and that the pairs (Φ_i, φ_i) ($i = 1, 2$) satisfy condition $(\varphi\gamma\delta)$ of Prop. 4.13 (i). Applying Prop. 4.13 (ii) (the existence part), for $\varsigma = \beta : F_1 \rightarrow F_2$ (the associativity 2-isomorphism of the \mathcal{G}_2 action on \mathcal{X}), we get $\beta_Y : \Phi_1 \rightarrow \Phi_2$ (the associativity 2-isomorphism of A_Y). The higher coherence $(xghl)$ (the pentagon) for β_Y follows from the one for β by applying the uniqueness part of Prop. 4.13 (ii) to the appropriate 2-isomorphisms that relate the five obvious maps $\mathcal{X} \times_{M_2} \mathcal{G}_2 \times_{M_2} \mathcal{G}_2 \times_{M_2} \mathcal{G}_2 \rightarrow \mathcal{X}$. □

Remark A.11 The proof of Lemma A.10 hides the explicit form of the action induced on the prequotient. At the level of objects, as already mentioned, we have that xg_2 in $\mathcal{X}/_p \mathcal{G}_1$ is the same as in \mathcal{X} . We now write how morphisms are multiplied. Let $[g_1, b] : x \rightarrow \bar{x}$ and $j_2 : g_2 \rightarrow \bar{g}_2$ be morphisms in $\mathcal{X}/_p \mathcal{G}_1$ and \mathcal{G}_2 respectively (above the same morphism in M_2). In particular, $b : xg_1 \rightarrow \bar{x}$ is a morphism in \mathcal{X} . Looking at the proof of Prop. 4.9 and at Remark 6.16, one can verify that

$$[g_1, b] \cdot j_2 = [g_1, (b \cdot j_2) \circ \delta]$$

where δ is defined in (A.3).

Lemma A.12 *There is a canonical isomorphism $(\mathcal{X}/_p \mathcal{G}_1)/_p \mathcal{G}_2 \xrightarrow{\sim} \mathcal{X}/_p \mathcal{G}$.*

PROOF: The universal property of the prequotient (Prop. 4.9) induces the maps Φ_1 and Φ in the diagram

$$\begin{array}{ccc}
\mathcal{X} & & \\
\downarrow & \searrow & \\
\mathcal{X}/_p \mathcal{G}_1 & \xrightarrow{\Phi_1} & \mathcal{X}/_p \mathcal{G} \\
\downarrow & \nearrow \Phi & \\
(\mathcal{X}/_p \mathcal{G}_1)/_p \mathcal{G}_2 & &
\end{array}$$

The canonical map Φ can be described explicitly and the following argument shows that it is an isomorphism. Note that, at the level of objects, all the arrows in the above diagram are identities, so that we just have to check that Φ is fully faithful. Let $x, y \in \text{Obj}(\mathcal{X})$. We consider triples (g_1, g_2, b) where $g_i \in \mathcal{G}_i$, $\mathbf{t}(g_i) = \mathbf{a}_i(x)$, and $b : (x \cdot g_2) \cdot g_1 \rightarrow y$ is a morphism in \mathcal{X} . We denote by $\xi_1 : x \cdot (g_1, g_2) \xrightarrow{\sim} (x \cdot g_2) \cdot g_1$ the isomorphism of the first line of (A.3). Associated with such a triple, we have a morphism $[g_2, [g_1, b]] : x \rightarrow y$ in $(\mathcal{X}/_p \mathcal{G}_1)/_p \mathcal{G}_2$, and a morphism $[(g_1, g_2), b \circ \xi_1] : x \rightarrow y$ in $\mathcal{X}/_p \mathcal{G}$. Given another triple $(\bar{g}_1, \bar{g}_2, \bar{b})$, one can check that $[g_2, [g_1, b]] = [\bar{g}_2, [\bar{g}_1, \bar{b}]]$ if and only if $[(g_1, g_2), b \circ \xi_1] = [(\bar{g}_1, \bar{g}_2), \bar{b} \circ \xi_1]$ (the condition for both equalities being the existence of $j_1 : g_1 \rightarrow \bar{g}_1$ and $j_2 : g_2 \rightarrow \bar{g}_2$ making the appropriate diagrams commute, compare with Eq. (4.3)). It follows that the formula defining the action of Φ on morphisms, given by

$$[g_2, [g_1, b]] \mapsto [(g_1, g_2), b \circ \xi_1],$$

is a bijection, hence Φ is fully faithful, as we wanted to verify. (One can check that the previous formula indeed gives the action of Φ by looking at the proof of Prop. 4.9.) \square

Lemma A.13 *If the \mathcal{G} -action on \mathcal{X} is 1-free (Def. 4.7), then:*

- (a) *The induced action of \mathcal{G}_1 on \mathcal{X} is 1-free.*
- (b) *The induced action of \mathcal{G}_2 on $\mathcal{X}/_p \mathcal{G}_1$ is 1-free.*

PROOF: The action of \mathcal{G} being 1-free means that $\text{id}_x \cdot (j_1, j_2) = \text{id}_x \cdot (j'_1, j'_2)$ implies that $j_1 = j'_1$ and $j_2 = j'_2$.

In order to prove that the \mathcal{G}_1 -action on \mathcal{X} is 1-free, assume that $\text{id}_x \cdot j_1 = \text{id}_x \cdot j'_1$. Using the definition of the induced action we have $\text{id} \cdot (j_1, 1_2) = \text{id}_x \cdot j_1 = \text{id}_x \cdot j'_1 = \text{id} \cdot (j'_1, 1_2)$, and the hypothesis of 1-freeness of the \mathcal{G} -action implies that $j_1 = j'_1$, as needed.

We now prove that the \mathcal{G}_2 -action on the prequotient $\mathcal{X}/_p \mathcal{G}_1$ is 1-free. Assume that $\tilde{\text{id}}_x \cdot j_2 = \tilde{\text{id}}_x \cdot j'_2$, where the tilde is used to remind us that $\tilde{\text{id}}_x = [1_1, \varepsilon_1]$ is an identity arrow in $\mathcal{X}/_p \mathcal{G}_1$ rather than \mathcal{X} . Here 1_1 and ε_1 are the appropriate identity (neutral element) and identity 2-isomorphism of \mathcal{G}_1 . Moreover, $j_2, j'_2 : g_2 \rightarrow \bar{g}_2$.

From Remark A.11 we see that $[1_1, (\varepsilon_1 \cdot j_2) \circ \delta] = \tilde{\text{id}}_x \cdot j_2 = \tilde{\text{id}}_x \cdot j'_2 = [1_1, (\varepsilon_1 \cdot j'_2) \circ \delta]$ as morphisms $xg_2 \rightarrow x\bar{g}_2$ in $\mathcal{X}/_p \mathcal{G}_1$. Hence there exists an isomorphism $j_1 : 1_1 \rightarrow 1_1$ in \mathcal{G}_1 such that the outer rectangle

$$\begin{array}{ccccccc}
(xg_2)1_1 & \xrightarrow{\xi_1} & x(1_1, g_2) & \xrightarrow{\xi_2} & (x1_1)g_2 & \xrightarrow{\varepsilon_1 \text{id}_{g_2}} & xg_2 \xrightarrow{\text{id}_x j_2} x\bar{g}_2 \\
(\text{id}_x \text{id}_{g_2})j_1 \downarrow & & \text{id}_x(j_1, \text{id}_{g_2}) \downarrow & & & & \text{id} \downarrow \\
(xg_2)1_1 & \xrightarrow{\xi_1} & x(1_1, g_2) & \xrightarrow{\xi_2} & (x1_1)g_2 & \xrightarrow{\varepsilon_1 \text{id}_{g_2}} & xg_2 \xrightarrow{\text{id}_x j'_2} x\bar{g}_2
\end{array}$$

commutes, where ξ_1, ξ_2 are defined by the first and second line of (A.3), respectively (one has $\xi_2 \xi_1 = \delta$). Since the left rectangle commutes (ξ_1 is a natural transformation) then the right one commutes as well. One can show that $(\varepsilon_1 \cdot \text{id}_{g_2}) \circ \xi_2$ is the identity of $x(1_1, g_2) = xg_2$ (one uses higher coherence $(x1g)$ of the original \mathcal{G} -action on \mathcal{X} , and $\rho_1 = \lambda_1 : 1_1 \cdot 1_1 \rightarrow 1_1$). As a result, the above right rectangle becomes the (commutative) triangle

$$\begin{array}{ccc} x(1_1, g_2) & & \\ \text{id}_x(j_1, \text{id}_{g_2}) \downarrow & \searrow \text{id}_x(\text{id}_{1_1}, j_2) & \\ x(1_1, g_2) & \xrightarrow{\text{id}_x(\text{id}_{1_1}, j'_2)} & x(1_1, \bar{g}_2) \end{array}$$

from which it follows that $\text{id}_x(j_1, j'_2) = \text{id}_x(\text{id}_{1_1}, j_2)$. By the assumption of 1-freeness of the original \mathcal{G} -action we conclude that $j_2 = j'_2$ (and $j_1 = \text{id}_{1_1}$), as needed. \square

Proposition A.14 *Assume that \mathcal{X} is a prestack, $\mathcal{G}_1, \mathcal{G}_2$ are stacky groupoids, and the \mathcal{G} -action on \mathcal{X} is 1-free, where $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$. Then there is a canonical isomorphism*

$$(\mathcal{X}/\mathcal{G}_1)/\mathcal{G}_2 \xrightarrow{\sim} \mathcal{X}/(\mathcal{G}_1 \times \mathcal{G}_2).$$

PROOF: Consider the diagram

$$\begin{array}{ccccc} (\mathcal{X}/\mathcal{G}_1)/_p \mathcal{G}_2 & \xleftarrow{\Psi} & (\mathcal{X}/_p \mathcal{G}_1)/_p \mathcal{G}_2 & \xrightarrow{\Phi} & \mathcal{X}/_p (\mathcal{G}_1 \times \mathcal{G}_2) \\ \downarrow \text{b} & & \downarrow \text{b} & & \downarrow \text{b} \\ (\mathcal{X}/\mathcal{G}_1)/\mathcal{G}_2 & \xleftarrow{\Psi^\#} & (\mathcal{X}/_p \mathcal{G}_1)/\mathcal{G}_2 & \xrightarrow{\Phi^\#} & \mathcal{X}/(\mathcal{G}_1 \times \mathcal{G}_2) \end{array}$$

where the vertical maps are stackifications. We first consider the right square. From Proposition 4.8 and Lemma A.13 it follows that $\mathcal{X}/_p \mathcal{G}_1$, $(\mathcal{X}/_p \mathcal{G}_1)/_p \mathcal{G}_2$ and $\mathcal{X}/_p (\mathcal{G}_1 \times \mathcal{G}_2)$ are prestacks. The map Φ is the isomorphism of Lemma A.12 (hence $\Phi^\#$ is an isomorphism as well). Let us look at the map Ψ . The stackification map $\mathcal{X}/_p \mathcal{G}_1 \rightarrow \mathcal{X}/\mathcal{G}_1$ is \mathcal{G}_2 -equivariant by Lemma A.9; from general facts, see Prop. 2.2 (iii), it is also a monomorphism and an epimorphism. We define Ψ to be the induced map between the prequotients by \mathcal{G}_2 (Proposition 4.13), and we use Proposition 4.16 to deduce that Ψ is a monomorphism and an epimorphism. Next, it follows from Lemma A.9 (and Prop. 4.8) that $(\mathcal{X}/\mathcal{G}_1)/_p \mathcal{G}_2$ is a prestack; hence, we can apply Prop. 2.2 (iv) to conclude that $\Psi^\#$ is an isomorphism. Finally, the desired canonical isomorphism is:

$$\Phi^\# \circ (\Psi^\#)^{-1} : (\mathcal{X}/\mathcal{G}_1)/\mathcal{G}_2 \xrightarrow{\sim} \mathcal{X}/(\mathcal{G}_1 \times \mathcal{G}_2).$$

\square

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